λ — BASES AND λ — NUCLEARITY

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
MOHD. AMIN SOFI

to the
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

MARCH, 1981

To

My Parents

Dr. P.K. Kamthan,
M.Sc., Ph.D., D.Sc.
Professor of Mathematics,
Indian Institute of Technology, Kanpur
Kanpur-208016 (India)

CERTIFICATE

This is to certify that the research work embodied in the present dissertation entitled "λ-Bases and λ-Nuclearity" by Mr. Mohd. Amin Sofi, a Ph.D. scholar of this Department, has been carried out under my supervision and that it has not been submitted elsewhere for any degree or diploma.

(P.K. Kamthan)

April - 1981

ACKNOWLEDGEMENTS

I take this opportunity to express my deepest sense of gratitude to my teacher and supervisor, Prof. P.K. Kamthan for suggesting this subject and for his continuous interest, invaluable advice and constructive criticism throughout the course of this work. His willingness to discuss the material at every stage despite his busy schedule and his ability to provide encouragement in moments of doubt, has gone a long way in the successful completion of this thesis.

I wish to express my sincere thanks to Professor P.K. Kamthan and Dr. Manjul Gupta of this Department for the stimulating seminars on functional analysis held under their leadership during 1979-80. I further wish to express my warmest thanks to Dr. Manjul Gupta for her useful comments, lucid suggestions and encouraging remarks at various stages of this work.

To my friends Messers Anwar, Chetty, Das, Deheri, Ishaq and Meraj, I owe my indebtedness for their assistance in several ways during the completion stage of this thesis. I am particularly grateful to Dr. Q.J.A. Khan who shared a lot of my burden while I was busy in compiling this thesis. My thanks are also due to my colleague Mr. John Patterson whose company through all these years provided an atmosphere congenial for carrying out this work.

My word of thanks also goes to Messers S.K. Tewari and G.L. Misra for their skilful typing and to A.N. Upadhya for his careful cyclostyling the thesis.

Mohd. Amin Sofi

April - 1981

CONTENTS

CHAPTER 0

PRELIMINARIES

1. Introduction : The material incorporated in this chapter is essentially meant to serve as a ready reference to the prerequisites of the work embodied in this dissertation. We present here a brief sketch of the concepts and results from the theory of (i) locally convex spaces, (ii) sequence spaces, (iii) Schauder bases, (iv) nuclear spaces and (v) several other miscellaneous notions and results that have been used in the sequel. All the results in this chapter are stated without proofs and can be found in any of the standard texts or monographs and research papers or theses; for instance one may look into [45], [58], [59], [65], [70], [89], [111] and [123] for further details. In the course of our work, we will use the following notation:

$\mathbb{N} = \{1,2,3,\ldots\}$.

$\mathbb{R}$ = Field of real numbers.

$\mathbb{C}$ = Field of complex numbers.

$\mathbb{K}$ = Scalar field $\mathbb{R}$ or $\mathbb{C}$.

$\mathbb{R}^+ = [0,\infty)$.

2. Locally Convex Spaces : In all that follows $X, Y, \ldots$ denote Hausdorff locally convex spaces (abbreviated hereafter

as l.c. TVS) over the field $\mathbb{K}$. Occasionally we shall also write X [resp. Y,...] as $(X,\mathcal{J})$ [resp. $(Y,\mathcal{G})$,...] whenever we wish to attach importance to the locally convex topology $\mathcal{J}$ [resp. $\mathcal{G}$,...] associated with X [resp. Y,...]. Also, the symbols E,F,... and H with or without suffix will stand respectively for normed spaces or a Hilbert space. The symbol $\mathcal{D}$ will stand for the collection of all continuous seminorms on X, generating the topology $\mathcal{J}$ of X. We denote by $\mathcal{U}$ a fundamental neighbourhood system at the origin consisting of barrels in X. A fundamental family of absolutely convex bounded sets in X will be denoted by $\mathcal{B}$. In order to emphasize the topology $\mathcal{J}$ or the space X involved, we shall find it convenient to denote these symbols by $\mathcal{D}_{\mathcal{J}}$ (or $\mathcal{D}_X$), $\mathcal{U}_{\mathcal{J}}$ (or $\mathcal{U}_X$) and $\mathcal{B}_{\mathcal{J}}$ (or $\mathcal{B}_X$). The members of $\mathcal{U}$ will be denoted by u,v,..., while the members of $\mathcal{D}$ will be designated by p,q,..., and those of $\mathcal{B}$ by A,B,... . Also for $u \in \mathcal{U}$, p_u will stand for the Minkowski functional on X, relative to u. The topological dual of an l.c. TVS X will be designated by X^*. The symbol $P(\mathbb{N})$ will stand for the set of all permutations on $\mathbb{N}$.

For l.c. TVS X and Y, we will use the symbol $L(X,Y)$ for the space of all continuous linear operators from X into Y. If $X = Y$, we write $L(X)$ for $L(X,Y)$. $F(X,Y)$ will stand for the class of all finite-dimensional operators $T \in L(X,Y)$ (i.e., $\dim T(X) < \infty$) while $F_n(X,Y)$ will designate

the subspace of $F(X,Y)$ consisting of operators $T \in L(X,Y)$ with $\dim T(X) < n$. Also, we write $F(X,X) = F(X)$ and $F_n(X,X) = F_n(X)$.

The proof of the following theorem due to Auerbach can be found in [89], p. 136.

<u>Theorem</u> 0.2.1 : Each mapping $T \in F(E,F)$ with $\dim T(E) = n$, can be represented in the form

$$T(x) = \sum_{i=1}^{n} \lambda_i \langle x,f_i \rangle y_i$$

with $f_i \in E^*$, $y_i \in F$, $||f_i|| \leq 1$, $||y_i|| \leq 1$, $1 \leq i \leq n$ so that the inequality

$$|\lambda_i| \leq ||T||$$

holds for each i, $1 \leq i \leq n$.

<u>Duality</u> : A pair of vector spaces X and Y over the same field $\mathbb{K}$ is said to be in <u>duality</u> provided there exists a bilinear form $B: X \times Y \to \mathbb{K}$, which separates points of both X and Y, i.e., for each $x \in X$ [resp. Y] there exists $y \in Y$ [resp. X] such that $B(x,y) \neq 0$. For each $y \in Y$, the function $q_y: X \to \mathbb{R}^+$, $q_y(x) = |B(x,y)|$, $x \in X$, defines a semi-norm on X. The topology $\sigma(X,Y)$ on X generated by the collection $\{q_y: y \in Y\}$ of these seminorms on X is called the <u>weak topology</u> defined by the dual pair $\langle X,Y \rangle$. The weak topology $\sigma(Y,X)$ on Y is likewise defined.

Given a collection $\mathcal{S}$ of $\sigma(Y,X)$-bounded subsets of Y, the $\mathcal{S}$-_topology_ on X (or _the topology of uniform convergence on sets in_ $\mathcal{S}$) is defined to be the locally convex topology generated by the collection $\{p_A : A \in \mathcal{S}\}$ of seminorms p_A on X, where

$$p_A(x) = \sup_{y \in A} |B(x,y)|, \quad x \in X.$$

If $\mathcal{S}$ consists of all $\sigma(Y,X)$-bounded [resp. balanced, convex $\sigma(Y,X)$-compact] subsets of Y, the corresponding $\mathcal{S}$-topology on X, denoted by $\beta(X,Y)$ [resp. $\tau(X,Y)$] is called the _strong topology_ [resp. _Mackey topology_] on X. In a similar fashion, the _strong * topology_ $\beta^*(X,Y)$ on X is the $\mathcal{S}$-topology generated by all $\beta(Y,X)$-bounded subsets of Y.

On the other hand, if X and Y are l.c. TVS over $\mathbb{K}$ and $\mathcal{S}$ is some collection of bounded sets in X, one can define an $\mathcal{S}$-topology on $L(X,Y)$ which is generated by the collection $\{p_{A,u} : A \in \mathcal{S},\ u \in \mathcal{U}_Y\}$ of seminorms $p_{A,u}$ on $L(X,Y)$ where

$$p_{A,u}(T) = \sup_{x \in A} p_u(Tx), \quad T \in L(X,Y).$$

$L(X,Y)$ endowed with an $\mathcal{S}$-topology will be denoted by $L_{\mathcal{S}}(X,Y)$. The $\mathcal{S}$-topology obtained by taking $\mathcal{S}$ to be the collection of all bounded (resp. precompact) subsets of X and denoted by β(resp. c), is called the _strong topology_ (resp. _topology of precompact convergence_) on $L(X,Y)$. It is readily seen that for each l.c. TVS X, one has $L_\beta(X,\mathbb{K}) = (X^*,\beta(X^*,X))$.

For an l.c. TVS X, the pair $\langle X,X^*\rangle$ forms a dual system via the canonical bilinear form: $\langle x,f\rangle = f(x)$, $x \in X$, $f \in X^*$. Thus X (and X^*) can be equipped with several polar topologies resulting from the dual pair $\langle X,X^*\rangle$. We are now in a position to state the following definitions:

Definition 0.2.2 : An l.c. TVS $(X,\mathcal{F})$ is said to be (i) barrelled if $\mathcal{F} = \beta(X,X^*)$; (ii) bornological if $\mathcal{F} = \tau(X,X^*)$ and each bounded linear functional on X is continuous; (iii) infrabarrelled if $\mathcal{F} = \beta^*(X,X^*)$; (iv) Mackey if $\mathcal{F} = \tau(X,X^*)$; (v) semi-reflexive (resp. reflexive) if $J:X \to X^{**}$ is surjective (resp. a topological isomorphism) where $[J(x)](f) = f(x)$, $x \in X$, $f \in X^*$ and $X^{**} = (X^{**},\beta(X^{**},X^*))$; (vi) S-space if $(X^*, \sigma(X^*,X))$ is sequentially complete; and (vii) dual metric if X has a fundamental sequence of bounded sets and that every $\beta(X^*,X)$-bounded sequence in X^* is equicontinuous.

We recall the following result from [45].

Theorem 0.2.3 (Banach-Steinhaus theorem) : Let X be a barrelled space and Y an l.c. TVS. Suppose $\{T_n\} \subset L(X,Y)$ such that $T(x) = \lim\limits_n T_n(x)$ exists for each $x \in X$. Then T is continuous.

Definition 0.2.4 : For a dual pair $\langle X,Y\rangle$, a locally convex topology $\mathcal{F}$ on X is said to be compatible with the dual pair $\langle X,Y\rangle$ if $(X,\mathcal{F})^* = Y$.

The following results on compatible topologies can be found in [45] and [111].

Theorem 0.2.5 (Mackey-Arens) : A locally convex topology $\mathcal{J}$ on X is compatible with the duality $\langle X,Y\rangle$ if and only if $\sigma(X,Y) \subset \mathcal{J} \subset \tau(X,Y)$.

Theorem 0.2.6 : For a dual pair $\langle X,Y\rangle$, all compatible topologies on X have the same bounded sets.

We also recall the following characterisation of finite dimensional topological vector spaces (cf. [45], p. 147):

Theorem 0.2.7 : A Hausdorff TVS X is finite dimensional if and only if it has a precompact neighbourhood at the origin.

For l.c. TVS X and Y, each $T \in L(X,Y)$ determines a linear map $T^*: Y^* \to X^*$ called the adjoint of T, which is defined by the equation

$$(T^*f)(x) = f(T(x)),\ x \in X,\ f \in Y^*.$$

We end up this section with the following result on the adjoint of a map:

Proposition 0.2.8 : Let $(X, \mathcal{J}_1)$ and $(Y, \mathcal{J}_2)$ be l.c. TVS. If $T:X \to Y$ is $\mathcal{J}_1$-$\mathcal{J}_2$ continuous then it is $\sigma(X,X^*)$-$\sigma(Y,Y^*)$-continuous. In this case $T^*:Y^* \to X^*$ exists and

is $\sigma(Y^*,Y)$- $\sigma(X^*,X)$-continuous. Conversely, if T is $\sigma(X,X^*)$-$\sigma(Y,Y^*)$ continuous, then it is $\tau(X,X^*)$-$\tau(Y,Y^*)$ continuous.

<u>The normed spaces associated with a locally convex space</u>: For an l.c. TVS X and $u \in \mathcal{U}_X$, we denote by $\hat{X}_u$ the completion of the normed space X_u obtained by taking the quotient of X relative to $\ker p_u$-the norm on X_u being given by the function $\hat{p}_u(x_u) = p_u(x)$, $x_u = x + \ker p_u$, $x \in X$. If $u,v \in \mathcal{U}_X$ with $v < u$ (u <u>absorbs</u> v), we obtain a canonical continuous map $\phi_u^v : X_v \to X_u$, where $\phi_u^v(x_v) = x_u$, $x_v \in X_v$. Also the canonical continuous surjection $X \to X_u$ is denoted by ϕ_u with $\phi_u(x) = x_u$, $x \in X$. The symbols $\hat{\phi}_u^v$ and $\hat{\phi}_u$ will respectively stand for the unique continuous extensions of ϕ_u^v and ϕ_u to the spaces $\hat{X}_v$ and $\hat{X}_u$. One obviously has $\hat{\phi}_u = \hat{\phi}_u^v \circ \hat{\phi}_v$. On the other hand, the normed space X_A associated with $A \in \mathcal{B}_X$ is the subspace of X spanned by A, which can be normed through the function $||x||_A = p_A(x) = \inf\{\lambda > 0 : x \in \lambda A\}$, $x \in X_A$. For $A,B \in \mathcal{B}_X$ with $A < B$, one has the natural continuous injections $i_A : X_A \to X$ and $i_{AB} : X_A \to X_B$, defined by $i_A(x) = x$ and $i_{AB}(x) = x$, $x \in X_A$, respectively. We have $i_A = i_B \circ i_{AB}$. The proof of the following theorem on the dual of X_u can be found in [89].

<u>Proposition</u> 0.2.9 : For $u \in \mathcal{U}_X$, the topological dual of $(X_u, \hat{p}_u)$ can be identified with the Banach space $(X^*(u^o), ||\cdot||_{u^o})$

under the map $\psi : X^*(u^o) \to (X_u)^*$ where $[\psi(a)](x_u) = \langle x,a \rangle$, $x_u = x + \ker p_u$, $a \in X^*(u^o)$.

3. <u>Sequence Spaces</u> : For the notions and results of this section we follow [58], (see also [65]). We assume throughout that ω will stand for the vector space of all scalar-valued sequences. By a <u>sequence space</u> we mean a subspace λ of ω such that $sp\{e^n; n \geq 1\} \subset \lambda$, where e^n denotes the nth <u>unit vector</u> $\{0,0,\ldots,0,1,0,\ldots\}$, 1 being placed at the nth co-ordinate. The symbols $\lambda^\times$ and λ^β will respectively stand for the <u>Köthe</u> (or α-) [resp. β-] dual of the sequence space λ :

$$\lambda^\times = \{ x \in \omega : \sum_{i \geq 1} |x_i y_i| < \infty, \forall\, y \in \lambda \};$$

$$\lambda^\beta = \{ x \in \omega : \sum_{i \geq 1} x_i y_i \text{ converges}, \forall\, y \in \lambda \}.$$

Also we find it convenient to introduce the symbol $\lambda^\times_+$ for the set $\{ x \in \lambda^\times : x_n \geq 0, \forall\, n \geq 1 \}$. For $x,y \in \lambda$, the notation $x \leq y$ will be used to mean that $|x_n| \leq |y_n|$ for each $n \geq 1$. Also, $x << y$ will be used when one has $|x_n| < |y_n|$, $n \geq 1$.

It is easily seen that the vector spaces λ and $\lambda^\times$ (resp. λ and λ^β) can be put into duality via the (canonical) bilinear form $\langle x,y \rangle \to \sum_{i \geq 1} x_i y_i$, $x \in \lambda$, $y \in \lambda^\times$ (resp. λ^β). Thus one can equip a sequence space λ with several polar

topologies resulting from the dual pair $\langle\lambda,\lambda^{\times}\rangle$ or $\langle\lambda,\lambda^{\beta}\rangle$. In addition to these polar topologies, there is a natural topology, namely the <u>normal topology</u> $\eta(\lambda,\lambda^{\times})$ on λ which is generated by the collection $\{p_y;\ y\in\lambda^{\times}\}$ of seminorms p_y on λ, where

$$p_y(x) = \sum_{i\geq 1} |x_i y_i|,\ x\in\lambda .$$

In fact, this topology is only a special case of what is called a normal topology on a sequence space λ. Following Rosier [103] (cf. also De Grande-De Kimpe [22]), we say that a locally convex topology $\mathcal{T}$ on a (normal) sequence space λ is <u>normal</u> provided it has a fundamental neighbourhood system at the origin consisting of normal sets. In particular, if S is a normal <u>topologizing family</u> [i.e. $N\in S \Longrightarrow$ normal hull $\hat{N}\in S$] for λ consisting of a saturated family of $\sigma(\lambda^{\times},\lambda)$-bounded subsets of $\lambda^{\times}$ which covers $\lambda^{\times}$, then the corresponding S-topology on λ is a normal topology. If S is taken to consist of normal hulls of singletons in $\lambda^{\times}$, we get the natural topology $\eta(\lambda,\lambda^{\times})$.

Further, a sequence space λ is said to be <u>perfect</u> if $\lambda=\lambda^{\times\times}$; also λ is called <u>normal</u> (resp. <u>monotone</u>) if $\ell^{\infty}\lambda\subset\lambda$ (resp. $m_0\lambda\subset\lambda$, m_0 being the sequence space generated by the set of zeroes and ones).

We recall the following results from the theory of sequence spaces (cf. [58], p. 79, 82, 136).

Proposition 0.3.1 : Sequential convergence in $(\lambda, \sigma(\lambda, \lambda^{\times}))$ is the same as in $(\lambda, \eta(\lambda, \lambda^{\times}))$.

Proposition 0.3.2 : λ is perfect if and only if $(\lambda, \sigma(\lambda, \lambda^{\times}))$ is sequentially complete.

Proposition 0.3.3 : The topology $\eta(\lambda, \lambda^{\times})$ is compatible with the dual pair $\langle \lambda, \lambda^{\times} \rangle$.

In our subsequent work, we confine mainly to the following type of sequence spaces. Indeed, let P be an arbitrary subset of ω such that (i) each $x \in P$ is positive, (ii) for each $n \in \mathbb{N}$, there exists $x \in P$ with $x_n > 0$, and (iii) for $x, y \in P$, there exists $z \in P$ with $x \leq z$, $y \leq z$. Such a set P is called a power set (or a Köthe set) and the sequence space

$$\Lambda(P) = \{x \in \omega : p_y(x) = \sum_{i \geq 1} |x_i y_i| < \infty, \ \forall\, y \in P\},$$

is called a Köthe space generated by P. If the topology on $\Lambda(P)$ is not emphasized, it will be assumed without further reference that $\Lambda(P)$ is endowed with its natural topology $\mathcal{F}_P$ generated by the family $\{p_y : y \in P\}$ of seminorms. Throughout, a Köthe set will be denoted by $P_0, P_1, P_2, \ldots$. We also recall that $\Lambda(P)$ is complete relative to its topology $\mathcal{F}_P$.

A Köthe space $\Lambda(P)$ is said to be a G_∞-space or a smooth sequence space of infinite type (resp. a G_1-space or a

<u>smooth sequence space of finite type</u>) provided P satisfies the following (additional) conditions:

(iv) For each $x \in P$, $0 < x_n \leq x_{n+1} \leq \cdots$ (resp. $0 \leq x_{n+1} \leq x_n \leq \cdots$).

(v) For each $x \in P$, there exists $y \in P$ such that $x_n^2 \leq y_n$ (resp. $x_n \leq y_n^2$), $n \geq 1$.

In this case we say briefly that P is a G_∞-(resp. G_1-) <u>Köthe set</u>.

Let now $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function which is also (i) <u>subadditive</u>, i.e., $\phi(x+y) \leq \phi(x) + \phi(y)$, $x,y \in \mathbb{R}^+$, (ii) <u>monotone</u>, i.e., $\phi(x) \leq \phi(y)$ for $x,y \in \mathbb{R}^+$ with $x \leq y$ and satisfies (iii) $\phi(0) = 0$. Corresponding to such a function ϕ, we define the sequence space ℓ_ϕ by

$$\ell_\phi = \{ x \in \omega : \sum_{n\geq 1} \phi(|x_n|) < \infty \}.$$

The sequence space $\Lambda(P;\phi)$ associated with ϕ and a power set P is defined by

$$\Lambda(P;\phi) = \{x \in \omega : \sum_{n\geq 1} \phi(|x_n|)\, a_n < \infty, \ \forall\, a \in P\}.$$

A numerical sequence $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots$ is called an <u>exponent sequence</u> if $\lim_{n\to\infty} \alpha_n = \infty$. It can be easily verified that the sets $P_1 = \{\{R^{\alpha_n}\}: R > 0\}$ and $P_2 = \{\{R^{\alpha_n}\}: R < 1\}$ of sequences are Köthe sets of infinite and finite type, respectively. The Köthe space $\Lambda(P_1)$

associated with P_1 is called a <u>power series space of infinite type</u> and is denoted by $\Lambda_\infty(\alpha)$. Similarly $\Lambda_1(\alpha)$ denotes the Köthe space $\Lambda(P_2)$ generated by P_2, and is called a <u>power series space of finite type</u>. For $\alpha_n = \log n$, the space $\Lambda_\infty(\alpha)$ yields the Fréchet space of <u>rapidly decreasing sequences</u> to be denoted hereafter by s, while for $\alpha_n = n$, we get the space of entire functions.

<u>Definition</u> 0.3.4 : A Köthe space $\Lambda(P)$ is said to be <u>stable</u> if $\Lambda(P) \times \Lambda(P) \cong \Lambda(P)$.

<u>Proposition</u> 0.3.5 ([32], [117]) : A G_∞-space $\Lambda(P)$ is stable if and only if for each $a \in P$, there exists $b \in P$ with $\sup_{n\geq 1} \frac{a_{2n}}{b_n} < \infty$. For power series spaces $\Lambda_\infty(\alpha)$ (and $\Lambda_1(\alpha)$) of infinite (and finite) type, this condition is equivalent to $\sup_{n\geq 1} \frac{\alpha_{2n}}{\alpha_n} < \infty$.

4. <u>Schauder Bases And Approximation Property</u> : We recall that a pair $\{x_n; f_n\}$ of sequences $\{x_n\} \subset X$ and $\{f_n\} \subset X^*$, X being an l.c. TVS, is called a <u>Schauder base</u>, provided each $x \in X$ has a unique expansion

$$x = \sum_{n\geq 1} f_n(x)\, x_n .$$

<u>Definition</u> 0.4.1 : A Schauder base $\{x_n; f_n\}$ is said to be

(a) <u>equicontinuous</u>, if for each $p \in \mathcal{D}_X$, there exists $q \in \mathcal{D}_X$ such that $\sup_{n\geq 1} \{p(x_n)|f_n(x)|\} \leq q(x)$, $x \in X$;

(b) absolute, if for each $p \in \mathcal{D}_X$ and $x \in X$,

$\sum_{n\geq 1} p(x_n)|f_n(x)| < \infty$;

(c) bounded multiplier, if for each $\alpha \in \ell^\infty$, the series $\sum_{n\geq 1} \alpha_n f_n(x)\, x_n$ converges for each $x \in X$;

(d) unconditional, if for each $\pi \in \mathcal{P}(\mathbb{N})$, the series $\sum_{n\geq 1} f_{\pi(n)}(x) x_{\pi(n)}$ converges for each $x \in X$;

(e) boundedly complete, provided the series $\sum_{n\geq 1} \alpha_n x_n$ converges for every $\alpha = \{\alpha_n\} \in \omega$ for which $\{\sum_{i=1}^{n} \alpha_i x_i\}$ is bounded in X, and

(f) shrinking, if $\{f_n; Jx_n\}$ is a Schauder base for $(X^*, \beta(X^*,X))$.

For the theory of Schauder bases in Banach spaces, we refer to [70] and [112] whereas for an elementary treatment of bases in l.c. TVS, one may consult [20] (cf. [59] for an exhaustive treatment of Schauder bases in topological vector spaces).

We recall the following result due to Cook [19] on Schauder bases in reflexive spaces:

Theorem 0.4.2 : Every Schauder base in a reflexive space is shrinking and boundedly complete. Conversely, an l.c. TVS possessing a Schauder base which is both shrinking and boundedly complete, is semi reflexive.

Corresponding to a Schauder base in an l.c. TVS X, the symbols δ and μ will denote, without further reference, the following sequence spaces:

$$\delta = \{\{f_n(x)\} ; x \in X\} ,$$

$$\mu = \{\{f(x_n)\} ; f \in X^*\} .$$

Note that δ and μ can be placed in duality with respect to the bilinear form

$$\langle\alpha,\beta\rangle \to \langle x,f\rangle \quad ,$$

where $\alpha \in \delta$, $\beta \in \mu$; $x \in X$, $f \in X^*$ and $\alpha = \{f_n(x)\}$, $\beta = \{f(x_n)\}$.

<u>Definition</u> 0.4.3 : A sequence space λ equipped with a locally convex topology $\mathcal{T}$ is said to possess the AK-<u>property</u> if for each $x \in \lambda$, $x^{(n)} \to x$ in $\mathcal{T}$ where $x^{(n)} = \sum_{i=1}^{n} x_i e^i$ is the nth section of x.

It can be easily verified that any sequence space λ is an AK-space when equipped with its weak or normal topology, i.e., the biorthogonal system $\{e^n; e^n\}$ forms a base for $(\lambda, \sigma(\lambda,\lambda^\times))$ or $(\lambda, \eta(\lambda,\lambda^\times))$. The same is true for any Köthe space $\Lambda(P)$ in its natural topology $\mathcal{T}_P$. However, it is a routine exercise to verify that $\{e^n; e^n\}$ is actually a Schauder base for $\Lambda(P)$. In fact, this statement is only a special case of the more general

<u>Proposition</u> 0.4.4 [22]: Under every compatible S-topology, λ is an AK-space. Thus $\{e^n; e^n\}$ forms a (Schauder) base for $(\lambda, \mathcal{T}_S)$.

Before we end up our discussion on bases, we recall the following

<u>Definition</u> 0.4.5 : A Schauder base $\{x_n; f_n\}$ in a Fréchet space X is said to be <u>regular</u> if there exists an increasing sequence of seminorms $\{p_k; k\geq 1\}$ generating the topology of X, such that

$$\frac{p_{k+1}(x_n)}{p_k(x_n)} \leq \frac{p_{k+1}(x_{n+1})}{p_k(x_{n+1})}, \quad \forall\, k, n \in \mathbb{N}.$$

For more details on regular bases, we refer to [27] and [29].

<u>Approximation Property In Locally Convex Spaces</u> : Here we recall a few definitions concerning approximation properties in locally convex spaces. For an exhaustive treatment of this subject, we refer to [68] and [111].

<u>Definition</u> 0.4.6 : An l.c. TVS X is said to possess the <u>approximation property</u> if $\mathcal{F}(X)$ is dense in $L_c(X)$.

<u>Definition</u> 0.4.7 : An l.c. TVS X is said to have the <u>bounded approximation property</u>, provided $\mathcal{F}(X)$ is sequentially dense in $L_c(X)$.

<u>Definition</u> 0.4.8 : An l.c. TVS X is said to possess the <u>strong finite dimensional decomposition</u>, provided there exists a sequence $\{T_n\} \subset L(X)$ with $T_n \circ T_m = \delta_m^n A_n$ and $r \in \mathbb{N}$

such that $\dim T_n(X) \leq r$ for each $n \in \mathbb{N}$ and $x = \sum_{n \geq 1} T_n(x)$, $x \in X$.

5. <u>Nuclear Spaces</u> : Our standard references for several notions and results of this section are [89] and [123] (cf. also [111] and [118]).

Let E and F be Banach spaces. We begin with

<u>Definition</u> 0.5.1 : An operator $T \in L(E,F)$ is said to be <u>nuclear</u> if there exist $\{\alpha_n\} \in \ell'$, $\{f_n\} \subset E^*$ and $\{y_n\} \subset F$ with $||f_n|| \leq 1$, $||y_n|| \leq 1$, $n \geq 1$ such that

$$T(x) = \sum_{n \geq 1} \alpha_n f_n(x)\, y_n, \quad x \in E.$$

<u>Definition</u> 0.5.2 : An l.c. TVS is said to be <u>nuclear</u> if to each $u \in \mathcal{U}_X$, there corresponds $v \in \mathcal{U}_X$, $v < u$ such that $\Phi_u^v : \hat{X}_v \to \hat{X}_u$ is nuclear.

<u>Definition</u> 0.5.3 : An l.c. TVS X is called <u>dual nuclear</u> (or <u>co-nuclear</u>) provided $(X^*, \beta(X^*,X))$ is a nuclear space.

The proofs of the following results can be found in [89] (see also [111]).

<u>Proposition</u> 0.5.4 : Every nuclear space is Hilbertizable, i.e., the topology of a nuclear space X can be considered to be generated by a family of seminorms which can be obtained from inner product functions. Hence for each $u \in \mathcal{U}_X$, $\hat{X}_u$ is a Hilbert space.

Proposition 0.5.5 : A metric or dual metric space is nuclear if and only if it is dual nuclear.

Proposition 0.5.6 : Every nuclear Fréchet space is Montel, hence reflexive.

We also recall the following result, known as the Grothendieck-Pietsch criterion for the nuclearity of a sequence space (see [58], p. 288).

Proposition 0.5.7 : A Köthe space $\Lambda(P)$ [resp. $(\lambda, \eta(\lambda,\lambda^{\times}))$] is nuclear if and only if for each $a \in P$ [resp. $\lambda_+^{\times}$], there exists $b \in P$ [resp. $\lambda_+^{\times}$] such that $\sum_{n\geq 1} a_n/b_n < \infty$.

For power series spaces, the above result can be restated in the form of

Proposition 0.5.8 : A power series space $\Lambda_{\infty}(\alpha)$ [resp. $\Lambda_1(\alpha)$] of infinite [resp. finite] type is nuclear if and only if for some [resp. each] $R > 1$, $\sum_{n\geq 1} R^{-\alpha_n} < \infty$.

We also state the following criterion for the nuclearity of a G_{∞}-space (see [116], p. 498) :

Proposition 0.5.9 : A G_{∞}-space $\Lambda(P)$ is nuclear if and only if for each $k \in \mathbb{N}$, there exist $\rho > 0$ and $a \in P$ such that

$$n^k \leq \rho\, a_n, \quad \forall\, n \geq 1.$$

6. Miscellaneous : In this section we briefly touch upon several definitions and notations not explained in the preceding

sections that have been used in the course of our work. For further details of these notions, we refer to the corresponding source cited appropriately before or after the particular definition or result. We start with

Definition 0.6.1 : Let X and Y be l.c. TVS. An operator $T \in L(X,Y)$ is said to be bounded [resp. precompact, compact] provided there exists $u \in \mathcal{U}_X$ such that $T(u)$ is bounded [resp. precompact, relatively compact] in Y. $L_b(X,Y)$ [resp. $L_t(X,Y)$, $L_k(X,Y)$] will denote the vector space of all bounded [resp. precompact, compact] operators from X into Y. Clearly, one has

$$L_k(X,Y) \subset L_t(X,Y) \subset L_b(X,Y).$$

Definition 0.6.2 : A linear operator $T:X \to Y$ is said to be absolutely summing ($T \in \pi(X,Y)$) provided $\sum_{n\geq 1} p_v(Tx_n) < \infty$, for each $v \in \mathcal{U}_Y$ and $\{x_n\} \subset X$ whenever $\sum_{n\geq 1} |\langle x_n,f\rangle| < \infty$, for each $f \in X^*$.

It readily follows that $\pi(X,Y) \subset L(X,Y)$. If $X = E$, $Y = F$ and G are normed spaces, we have the following theorem due to Pietsch [89], p. 66.

Theorem 0.6.3 : The product $TS \in L(E,F)$ of mappings $S \in \pi(E,G)$ and $T \in \pi(G,F)$ yields a nuclear operator.

We say that an l.c. TVS X has the 'Dvoretzky-Rogers property' if each unconditionally convergent series in X is

absolutely convergent. As a direct consequence of Theorem 0.6.3, we get the following famous theorem of Dvoretzky and Rogers.

Theorem 0.6.4 : A normed space is finite dimensional if and only if it has the Dvoretzky-Rogers property.

For the following definitions, notations and related results, we follow [89].

Definition 0.6.5 : Let X be an l.c. TVS and $u, v \in \mathcal{U}_X$ with $v < u$. Then we have

(i) The nth Kolmogorov diameter of v relative to u is defined by $\delta_n(v,u) = \inf \{ \rho > 0 : v \subset \rho u + L ; \dim L < n \}$.

(ii) ϵ-capacity ($\epsilon > 0$) of v relative to u is defined by $M_\epsilon(v,u) = \sup \{ n \in \mathbb{N} : \exists\ x_1, x_2, \ldots, x_n \in v : x_i - x_j \notin \epsilon u,\ i \neq j \}$.

(iii) Order of v relative to u is defined by

$$\rho(v,u) = \limsup_{\epsilon \to 0} \frac{\log \log M_\epsilon(v,u)}{\log 1/\epsilon} .$$

Definition 0.6.6 ([114]) : The diametral dimension of an l.c. TVS X is defined to be the collection of all sequences $x = \{ x_n \} \in \omega$ such that to each $u \in \mathcal{U}_X$, there corresponds $v \in \mathcal{U}_X$, $v < u$ with $\lim_n x_n \delta_n(v,u) = 0$.

Definition 0.6.7 ([73], [89]) : The approximative dimension of an l.c. TVS X is defined by

$$\Phi(X) = \{f:\mathbb{R}^+ \to \mathbb{R}^+ : \forall\, u \in \mathcal{U}_X, \exists\, v \in \mathcal{U}_X,\ v < u \text{ s.t. } \lim_{\varepsilon \to 0} (f(\varepsilon))^{-1} M_\varepsilon(v,u) = 0\}.$$

Definition 0.6.8 ([89]): For an operator $T \in L(E,F)$, E and F being Banach spaces, we define the nth-approximation number $\alpha_n(T)$ and the nth-Kolmogorov number $\delta_n(T)$ by

$$\alpha_n(T) = \inf\{||T-A||: A \in F_n(E,F)\}$$

and

$$\delta_n(T) = \delta_n(T(U),V), \qquad n \in \mathbb{N},$$

U and V being the closed unit balls in E and F, respectively.

We remark that (cf. [114], p. 69) $\delta_n(v,u) = \delta_n(v(u))$ where $v(u)$ denotes the bounded subset of X_u given by $v(u) = \{x + \ker p_u : x \in v\}$. The relationship between $\alpha_n(T)$ and $\delta_n(T)$ is contained in

Proposition 0.6.9 : For $T \in L(E,F)$, we have

(i) $\delta_n(T) \leq \alpha_n(T) \leq n\, \delta_n(T)$, $n \geq 1$.

(ii) If T is compact and E and F are Hilbert spaces, then $\alpha_n(T) = \delta_n(T) = \alpha_n(T^*) = \delta_n(T^*)$, $n \geq 1$.

It has been observed by Hutton [46] that in (i) above, the inequality on the right hand side can be further improved as $\alpha_n(T) \leq (n^{1/2})\, \delta_n(T)$, $n \geq 1$. Also, it follows from [47]

that the identity $\alpha_n(T) = \alpha_n(T^*)$ in (ii) is valid in the more general case when E and F are taken to be arbitrary Banach spaces.

The proof of the following theorem known as the spectral decomposition theorem for compact operators, can be found in [68]. See also [89], p. 129.

Theorem 0.6.10 : Let H_1 and H_2 be Hilbert spaces and $T: H_1 \to H_2$ be a compact operator. Then there exist orthonormal sequences $\{e_n\}$ and $\{h_n\}$ in H_1 and H_2, respectively, as well as a sequence $\{\lambda_n\}$ with $\lambda_n > 0$ and $\lambda_n \to 0$ such that

$$T(x) = \sum_{n \geq 1} \lambda_n \langle x, e_n \rangle h_n, \; x \in H_1.$$

Theorem 0.6.11 ([89]) : For each compact operator $T: H_1 \to H_2$, one has

$$\alpha_n(T) = \lambda_n \; , \; n \geq 1$$

where $\{\lambda_n\}$ is some sequence of scalars appearing in the spectral decomposition of T.

We conclude this section with the following result due to Tikhomirov. See [114], p. 58.

Theorem 0.6.12 : Let E be a normed linear space and L a subspace of E with $\dim L = n$. Then for every bounded subset A of E, the inclusion $\delta(U \cap L) \subset A$ implies that $\delta_n(A) \geq \delta$. Here $\delta_n(A) = \delta_n(A, U)$.

Operator ideals : The concept of an A-space as introduced by Pietsch [90] in order to unify the notions of nuclear spaces, Schwartz spaces and other kinds of locally convex spaces, depends upon the concept of an operator ideal which is defined in

Definition 0.6.13 : A set A consisting of a certain collection of operators between Banach spaces is called an operator ideal provided it satisfies the following conditions:

(i) $A(E,F) = L(E,F) \cap A$, for all Banach spaces E and F;

(ii) $Q \circ T \circ S \in A(E,H)$ whenever $T \in A(F,G)$ and $S \in L(E,F)$, $Q \in L(G,H)$, for all Banach spaces E,F,G and H, and

(iii) $T, S \in A(E,F) \Longrightarrow T + S \in A(E,F)$.

A is called a semi-ideal of operators provided it satisfies (i) and (ii).

Definition 0.6.14 : An l.c. TVS X is said to be an A-space (A being an operator ideal) provided for each $u \in \mathcal{U}_X$, there exists $v \in \mathcal{U}_X$, $v < u$ such that $\hat{\phi}_u^v \in A(\hat{X}_v,\hat{X}_u)$.

For an elementary treatment on operator ideals and their applications to λ-nuclearity, one may look into [79] and several references cited therein.

CHAPTER 1

HISTORY AND MOTIVATION

Part I

Nuclear Spaces And Nuclear Operators

1. Development Of Nuclear Spaces : Nuclear locally convex spaces were introduced by Grothendieck around the year 1953 in his celebrated dissertation [40] wherein he laid the foundations of the theory of topological tensor products of locally convex spaces. Needless to say, the idea of a nuclear space was mainly motivated by his investigations into the problem of representing operators by means of kernels. This led him to define nuclear spaces through the equivalence of suitable π-and ε-tensor products. One also finds that the most important part of the theory of nuclear spaces is already contained in the fundamental work of Grothendieck referred to above; however, the machinery of topological tensor products which is used there to prove several results, is indeed very hard and cumbersome!

It was in the early sixties that A. Pietsch undertook the task of constructing a theory of nuclear spaces without resorting to tensor product techniques. In doing so, he first developed a theory of summability of families and sequences in locally convex spaces, and in the process, he

introduced several locally convex spaces of vector-valued sequences consisting of summable and absolutely summable sequences which were subsequently utilized in characterising nuclear spaces (see [86]). One of these results on the characterisation of nuclear spaces contains as a special case an elegant result of Grothendieck which says that the nuclearity of a Fréchet space is equivalent to its 'Dvoretzky-Rogers property'. This result suggests that the nuclearity of a locally convex space can be amply reflected through the behaviour of the associated generalized sequence spaces. It seems that this aspect of the theory of nuclear spaces has provided much of the motivation for an independent investigation of generalized sequence spaces, enshrined in the works of Pietsch [85], Gregory [39], Rosier [103] and De Grande-De Kimpe [22] (cf. also [17], [25], [41], [42] and [43] for further developments in this direction).

The theory of nuclear spaces attained further significance especially after Mityagin's work [72], [73] came to limelight. Indeed, inspired by a problem of Gelfand, he made use of the notions of nth-diameter [60] and ϵ-entropy [62] of Kolmogorov, diametral dimension of Bessaga, Pelczyński and Rolewicz [11] and the approximative dimension of Kolmogorov [61] and Pelczyński [83], to give several characterisations of nuclear spaces. Notable among these results are the following:

(I) An l.c. TVS X is nuclear if and only if to each $u \in \mathcal{U}_X$, there corresponds $v \in \mathcal{U}_X$, $v < u$ such that $\sum_{n \geq 1} \delta_n(v,u) < \infty$.

(II) A metric or dual metric space X is nuclear if and only if $\rho(K,u) = 0$, for each compact set K in X and $u \in \mathcal{U}_X$.

(III) An l.c. TVS X is nuclear if and only if for some (resp. each) $\alpha > 0$, $\{n^\alpha\} \in \Delta(X)$.

(IV) An l.c. TVS X is nuclear if and only if for some (resp. each) $\alpha > 0$, $[\exp(\epsilon^{-\alpha})] \in \Phi(X)$.

Further contributions along these lines have also been made by Schock [110] and Fenske and Schock [35]. A result which lends substance to the fact that the diametral dimension of a locally convex space is indeed a measure of its nuclearity, was established by Fenske and Schock in [36] where they prove that the class of locally convex spaces having the maximal diametral dimension forms a 'stability class' of nuclear spaces. We also mention the important work of Terzioğlu [114] where the problem of diametral dimension has been subjected to further investigations and subsequently utilized in exploring the conditions for the nuclearity and the Schwartz property of Köthe sequence spaces. For further details on Schwartz spaces and historical remarks on this subject matter, we refer to [51].

The first universality theorem on nuclear spaces was proved by T. and Y. Kōmura [63] in 1965. They proved that every nuclear space can be embedded as a subspace of a suitable topological product of s, the nuclear space of rapidly decreasing sequences - thus substantiating a conjecture of Grothendieck [40]. This result has been further improved by Saxon and Valdivia in [105] and [119], respectively, where the space s has been replaced by an arbitrary infinite dimensional Banach space. A recent work of Fehr and Jarchow [34] containing as a special case the Kōmura-Kōmura theorem and several other results on the imbedding of specific locally convex spaces, also deserves special mention.

It goes without saying that there has been considerable activity going on in recent years on various problems related to the theory of nuclear spaces. One of the most important problems in this direction that has aroused a fair amount of interest, is concerned with questions on the structure of nuclear Fréchet spaces. In essence this problem deals with such questions as to when a nuclear Fréchet space is isomorphic to a subspace, quotient space or a complemented subspace of a given nuclear Fréchet space. Instead of embarking upon a detailed discussion on the development of this problem, we refer to the excellent monograph of E. Dubinsky [29] for more details and references quoted therein.

2. <u>Schauder Bases And Nuclear Spaces</u> : The theory of Schauder bases in general locally convex spaces has proved particularly rich when applied to nuclear spaces. In this connection several deep and interesting results have been obtained by analysts, thus reflecting a strong connection between Schauder bases and nuclear spaces. For an exhaustive treatment of Schauder bases and the details of their relationship with nuclearity, we refer to Chapter 2 of a forthcoming monograph of Kamthan and Gupta [59].

The first result concerning the impact of nuclearity on Schauder bases in locally convex spaces, was proved by Dynin and Mityagin [33] in 1960. This result which is of fundamental importance in the structure theory of nuclear spaces, states that every Schauder base in a nuclear Fréchet space is absolute-thereby establishing the profound fact that nuclear Fréchet spaces with bases are essentially (nuclear Fréchet) Köthe spaces. This fact together with a result of Wojtyński [124] on Schauder bases in non-nuclear Fréchet spaces leads to the following characterisation of nuclear Fréchet spaces with bases:

(V) A Fréchet space X with a Schauder base is nuclear if and only if every base in X is absolute.

On the other hand, Pietsch [88] in 1966 while investigating absolute bases in nuclear Fréchet spaces, discovered

that the nuclearity of a Fréchet space X having an absolute base, is equivalent to the absoluteness of the dual base in $(X^*,\beta(X^*,X))$. This result was further extended to (LF)-spaces by Bennet and Cooper in [7]. Another important result giving a useful criterion for the nuclearity of a Fréchet space with a Schauder base, is contained in the works of Grothendieck [40], Pietsch [88] and Dynin and Mityagin [33]. Its refined version to the case of an arbitrary locally convex space possessing an equicontinuous base, was established by Terzioğlu [115], Kalton [50] and Kamthan [52]. It states that an l.c. TVS X having an equicontinuous base $\{x_n; f_n\}$ is nuclear if and only if to every $p \in \mathcal{D}_X$, there corresponds $q \in \mathcal{D}_X$, $q \geq p$ satisfying $\sum_{n \geq 1} \frac{p(x_n)}{q(x_n)} < \infty$. Further results in this direction have also been proved by Schock [109] and Rosenberger and Schock [102], followed by an important work of Kalton [50] where the author carries out a neat dissection of the concept of an absolute basis. Along with many other results, the author also establishes the aforementioned result of Grothendieck and Pietsch via the use of his new techniques which are quite different from and simpler than those of these authors. We also mention the work of Kamthan and Gupta [56]; [57] where, besides laying down results on the interrelationship between several notions of absolute bases as introduced by Kalton, they construct examples and counter-examples to justify further study of these concepts. Several recent

contributions due to De Grande-De Kimpe [25] and Ceitlin [18] need also be specifically mentioned.

On the other hand, Dragilev [27] in 1971 introduced the notion of a regular base in a nuclear Fréchet space and thus initiated a new direction of research into the problems related to nuclear power series spaces. Apart from the fact that regular bases turn out to be particularly useful in nuclear Fréchet spaces where it makes it easy to compute the Kolmogorov diameters of neighbourhoods, they possess the remarkable property that a nuclear Fréchet space having a regular base, has the 'quasi equivalence property', i.e., for any two bases $\{x_n\}$ and $\{y_n\}$ in X, there exist a sequence $\{r_n\}$ of positive numbers, permutations π and $\sigma \in P(\mathbb{N})$ and an isomorphism $J:X \to X$ such that $J(x_{\pi(n)}) = r_n y_{\sigma(n)}$, $n \geq 1$. Roughly speaking, this means that there are not too many bases in nuclear Fréchet spaces with a regular base. A useful generalization of this result has been obtained by Alpseymen [1]. For more details and references, we refer to an important work of Bessaga [8].

For a long time it has been an open problem whether every nuclear Fréchet space (which is always separable) has a basis. This problem which was posed by Grothendieck [40] in 1955, was solved in the negative by Mityagin and Zobin [75], [125] in 1974. Indeed, they constructed a nuclear

Fréchet space without a base. Their construction of such a space though hard and cumbersome, was considerably simplified by Djakov and Mityagin [26] in 1976. This simplification enabled them to produce nuclear Fréchet spaces of maximal diametral dimension, which do not admit even a strong finite dimensional decomposition! C. Bessaga [9] and B.S. Mityagin [74] further modified and simplified the Djakov-Mityagin arguments to clarify the two-dimensional intuition of this construction and assert the existence of nuclear Fréchet spaces which do not admit a structure weaker than that given by a basis. This was followed by the works of Dubinsky [28] and Dubinsky and Mityagin [30] where, employing the techniques of Djakov and Mityagin and the embedding techniques of Dubinsky, they prove that each nuclear Fréchet space not isomorphic to ω , admits a subspace and a quotient space without a base. It was, however, observed by Bessaga and Dubinsky [10] that a further clarification of the Djakov-Mityagin arguments and the embedding methods of Dubinsky, leads to much simpler and unified proofs of the aforementioned results of Dubinsky and Mityagin. Recently, Moscatelli [78] inspired by a problem of Dubinsky, established the existence of a nuclear Fréchet space without a continuous norm and which is not embeddable as a subspace of a countable product of Fréchet spaces, each admitting a continuous norm! It turns out that such a space provides one of the simplest examples of a nuclear Fréchet space without a base.

A problem closely related to the basis problem is that of Grothendieck's approximation problem, introduced in [40] which incorporates a wealth of information on approximation property in locally convex spaces. It turns out that every nuclear (Fréchet) space possesses the approximation property [111] (cf. also [59]) but not the bounded approximation property. A counter-example to this effect has been constructed by Dubinsky in 1978 and appears in [29]. For a detailed and comprehensive account of this subject, we refer to Dubinsky's monograph [29] which contains more or less all that is known about approximation properties in nuclear Fréchet spaces.

3. <u>Order Structure And Nuclear Spaces</u> : Although a good deal of literature is available on the diametral and approximative dimension of nuclear spaces, not much progress has been made concerning the order structure in nuclear spaces. In this context the only significant contribution seems to be that of Kōmura and Koshi [64]. Their results show that an order structure on a nuclear space sometimes leads to very deep and far reaching consequences, regarding the structure of the underlying locally convex space. One of these results depicting such a situation consists in the assertion that every nuclear (F)-lattice has a Schauder base. Besides, a characterisation of nuclearity in Fréchet spaces having a Schauder base has also been obtained by Lazar and Retherford [69]- thus reflecting a nice interplay between the order and

nuclear structure of a locally convex space. A detailed account of this aspect of the theory of nuclear spaces is already contained in the unpublished lecture notes of Bessaga and Retherford [12].

4. <u>Nuclear Operators</u> : Nuclear spaces as introduced by Grothendieck through tensor products can be conveniently expressed in terms of a class of continuous linear operators, known as nuclear operators, of which we outline a brief survey and trace their historical development in this subsection and follow the survey article of Kamthan [53].

The origin of nuclear operators can be traced back to 1936 when Schatten [106] and Schatten and von Neumann [108] came across these operators in their work on problems related to the mathematical foundations of Quantum Mechanics and called them the " trace class operators" . Realizing the importance of these operators, Grothendieck extended the notion of nuclear operators on Hilbert spaces- as was already done by Schatten and von Neumann - to the more general setting of Banach spaces and carried out a thorough investigation into the theory and structure of these operators in his dissertation [40]. Utilizing the notion of an absolutely summing operator [87], Pietsch was able to further simplify the theory of nuclear operators (and spaces) founded earlier by Grothendieck. Among several other results based on these techniques, he could

establish one of the deepest results in operator theory which leads to the profound conclusion that the class of nuclear spaces can be equivalently defined by means of the ideal of absolutely summing operators!

The first work after the fundamental paper of Mityagin [73], covering a theory of nuclear spaces and nuclear operators on Hilbert spaces, appeared in 1958. This treatise constitutes the fourth volume [38] of their monumental work on generalized functions, extending over a series of six volumes, by I.M. Gelfand and his collaborators. This work whose English translation later appeared in 1966, contains a wealth of information on the structure of nuclear spaces in the setting of countably Hilbert spaces. Also included in this work are several deep results concerning the metric order of sets in nuclear spaces. This was followed by Pietsch's monograph in 1965 and then by its English translation [89] in 1972, which incorporates all the basic and important results on nuclear operators and spaces, known till then. A recent book by Wong [123] which embodies - apart from the theory of nuclear spaces- the theory of Schwartz spaces and tensor products, is also a useful addition to the stock of literature on nuclear spaces. Another recent monograph by Dubinsky [29] and a forthcoming book by Mityagin, dealing with the structure of nuclear Fréchet spaces, are also worthy of special mention. Another useful source highlighting the diameter-approach to

nuclear (sequence) spaces, is a recent monograph of Kamthan and Gupta [58] on sequence spaces and series. We also mention the works of Shaefer [111] and Treves [118] where a brief account of the theory of nuclear operators and nuclear spaces has been incorporated.

The importance and significance of nuclear spaces lay mainly in the fact that with a few exceptions, all infinite dimensional locally convex spaces encountered in analysis are either normed spaces or nuclear spaces. Nuclear spaces, in particular include several important spaces known in analysis, for instance, the spaces of (i) analytic, (ii) entire, (iii) harmonic, (iv) infinitely differentiable functions and (v) distributions of L. Schwartz. Pietsch has rightly remarked that " if a theory of structure for locally convex spaces can be developed at all, then it must certainly be possible for nuclear spaces because they are more closely related to finite-dimensional spaces than are normed spaces". In view of these remarks and the privileged position occupied by nuclear spaces in analysis, it is not surprising that mathematicians have already made a beginning to construct generalizations or other kinds of variation of this theory.

Part II

Generalizations Of Nuclear Spaces

1. Development of λ-Nuclearity : We have already remarked that nuclear spaces can be defined by means of the class of nuclear operators in a natural way. A close examination of the concept involved in the definition of a nuclear operator reveals that these operators depend upon the Banach sequence space ℓ^1. It is, therefore, quite natural to consider the situation when ℓ^1 is replaced by an arbitrary sequence space λ. This leads to the notion of λ-nuclear operators which are used to define λ-nuclear spaces. Unlike the class of nuclear spaces where a rich and powerful theory is already available on the class of operators defining these spaces, no such corresponding theory exists for a thorough investigation of λ-nuclear spaces. However, the only significant contribution in this direction seems to be that of Dubinsky and Ramanujan [32] followed by that of Walker [122] where a detailed study of λ-nuclear and λ-summing operators has been carried out. Nevertheless, in spite of the absence of such a theory, there seems to have generated a great amount of interest and activity in this direction during the past few years, resulting in considerable development of the subject (cf. also a recent work of Kamthan [54]). In what follows, we trace a brief development of this subject in its chronological order and follow [53].

The first application of the idea of replacing ℓ^1 by a sequence space λ is due to Martineau [71] who in 1964 considered the strongly nuclear spaces defined by means of the space s of rapidly decreasing sequences. After a few years, Persson and Pietsch [84] considered the case when $\lambda = \ell^p$, $1 < p < \infty$ and showed that only ℓ^p-nuclear operators be studied as ℓ^p-nuclear spaces are the same as nuclear spaces. Almost simultaneously, Brudovski [15], [16] rediscovered the ideas of Martineau and along with many other results made an error that led Köthe to introduce the concept of uniform (s)-nuclearity for a Köthe sequence space. This work was followed by that of Spuhler [113] and Ramanujan [91] where these authors have investigated the problem of $\Lambda_\infty(\alpha)$-nuclearity, $\Lambda_\infty(\alpha)$ being a nuclear power series space of infinite type. Around the same period, further contributions were also made by Hogbe-Nlend [44] and De Grande-De Kimpe [21].

The first serious attempt to bring a unified theory of λ-nuclear spaces was made by Dubinsky and Ramanujan in [32] where a sufficient motivation is provided to develop a theory of λ-nuclear spaces. In extending the notion of $\Lambda_\infty(\alpha)$-nuclearity, they replace the power series space $\Lambda_\infty(\alpha)$ by a nuclear G_∞-space $\Lambda(P)$ where P is assumed to be countable. However, the study of $\Lambda(P)$-nuclearity associated with an arbitrary nuclear G_∞-space $\Lambda(P)$ was carried out by Terzioĝlu

in his important paper [116], where it is proved that there exists a duality between nuclear smooth sequence spaces of finite and infinite type. The expected development - during this period - of λ-nuclearity when λ is a smooth sequence space of finite type, is comparatively of less significance. The only contribution in this direction is that of Robinson [98] where the author has investigated the $\Lambda_1(\alpha)$-nuclearity, associated with a nuclear power series space $\Lambda_1(\alpha)$ of finite type. Later in 1975, the same author considered the λ-nuclearity associated with a regular nuclear Köthe sequence space λ. This was done in [99] where he also obtained the Grothendieck-Pietsch-Köthe criterion for this type of nuclearity. On the other hand, B. Rosenberger carried out his investigations on ϕ-nuclear spaces obtained by taking λ to be the sequence space ℓ_ϕ defined by means of a suitable function ϕ on $\mathbb{R}^+$.

In 1975, Ramanujan and Terzioǧlu [95] studied the $\Lambda_k(\alpha)$-nuclear spaces, corresponding to a power series space $\Lambda_k(\alpha)$ of radius of convergence k. Their results showed that the results on nuclearity associated with these spaces, though in marked contrast with those known about $\Lambda_1(\alpha)$-nuclearity, exhibit a strong resemblance with several results on $\Lambda(P)$-nuclearity, considered earlier by Terzioǧlu. Using the space $\Lambda_k(\alpha)$, they also introduce the notion of $\Lambda_{\mathbb{N}}(\alpha)$-nuclearity for locally convex spaces which was subsequently extended by

Ramanujan and Rosenberger [93] to $\Lambda(P;\mathbb{N})$-nuclearity. (It may be noted that $\Lambda_{\mathbb{N}}(\alpha)$ or $\Lambda(P;\mathbb{N})$ is not a sequence space). In this work they also investigate the $\hat{\Lambda}(P;\phi)$-nuclearity (of spaces) associated with a countable, monotone, nuclear G_∞-set P and a suitable function ϕ on $\mathbb{R}^+$. Besides, they prove that for a wide class of functions ϕ, the study of $\Lambda(P;\phi)$-nuclearity reduces to that of $\Lambda(Q)$-nuclearity, for some nuclear G_∞-set Q. The concept of $\Lambda(P;\mathbb{N})$-nuclearity has recently been exploited to investigate the structure of certain nuclear Fréchet spaces. Indeed, using this idea, Alpseymen [2] has characterized all subspaces with basis of the nuclear Fréchet space $L_f(a,\infty)$ of Dragilev, followed by a recent work of Apiola [6] where he has given a complete characterization of all subspaces of $L_f(a,\infty)$ (cf. also [120]).

On the other hand, an entirely different concept of nuclearity was considered by Joichi in [49] where (Z,λ)-nuclear mappings have been introduced, using a Banach space Z and a sequence space λ. For $\lambda = \ell^1$, these operators have been utilized in a natural way to define Z-nuclear spaces. In this connection, the aforementioned work seems to be the only contribution on Z-nuclear spaces and not much is known about this type of nuclearity.

Concerning the embedding of λ-nuclear spaces, positive results have been obtained in this direction by Rosenberger [101], Ramanujan and Terzioğlu [95] and Ramanujan and

Rosenberger [93] for several kinds of λ-nuclear spaces, associated with specific sequence spaces λ . A unified treatment of some of these results has recently appeared in an important work of Fehr and Jarchow [34] whereas Moscatelli in [77] has considered the problem of characterising those sequence spaces which admit a universal λ-nuclear Fréchet space.

2. λ-<u>Nuclearity</u> <u>And</u> λ-<u>Bases</u> : In view of the absolute basis theorem of Dynin and Mityagin [33] and that of Pietsch [88] on absolute bases in nuclear Fréchet spaces, one finds that in the theory of Schauder bases in locally convex spaces, the absolute bases are particularly well adapted in the study of nuclear Fréchet spaces. Also it turns out that the notion of absolute bases is closely related to the sequence space ℓ^1 in much the same way as nuclear operators depend on this space. It is, therefore, reasonable to extend the notion of absolute basis to λ-basis, corresponding to an arbitrary sequence space λ and then investigate the problems related to λ-bases and λ-nuclear spaces, motivated by known results from the theory of absolute bases and nuclear spaces. In this direction, the first attempt was made by Schock [109] in 1969. Here he introduced the notion of a p-absolute basis, $p > 1$ and obtained several results analogous to those known from the theory of absolute bases. This concept was further investigated by Kalton in [50] where he improved upon certain

results contained in the above-mentioned work of Schock. One of these results, displaying a strong connection of p-absolute bases with nuclearity, is the following:

(VI) A Fréchet space with a p-absolute basis and a (different) q-absolute basis with $p \neq q$, is nuclear.

Another generalization of this kind to ϕ-absolute bases, was carried out by Rosenberger and Schock [102] and among several other interesting results, they also proved an analogous version of the Dynin-Mityagin theorem for Schauder bases in ϕ-nuclear Fréchet spaces. On the other hand, an entirely different situation ensued when mathematicians considered λ-bases associated with a nuclear sequence space λ. In this direction, Dubinsky and Ramanujan [31] in 1972 considered the notion of $\Lambda_\infty(\alpha)$-bases where $\Lambda_\infty(\alpha)$ is a nuclear power series space of infinite type. Their results showed a strong deviation from the classical theory of bases in nuclear spaces.

A marked progress toward investigating λ-bases was made in 1976 by De Grande-De Kimpe in her work [24]. Besides deriving certain results concerning the impact of a λ-basis on the structure of the underlying space, she also investigated the weak basis problem for λ-bases. The last section of this paper deals with the problems guarenteeing conditions for the λ-nuclearity of a locally convex space having a λ-base.

The basis problem for λ-nuclear spaces was first solved by Mityagin and Zobin [75] for the case when $\lambda = \ell^1$, as mentioned earlier. The solution to this problem for $\Lambda(P)$-nuclear Fréchet spaces was given by Mori in [76]. However, a slight modification of the techniques of Djakov and Mityagin [26], led Ramanujan and Rosenberger [94] to assert that in any class of λ-nuclear spaces where λ is a normal sequence space, one can find a Fréchet space which does not have a basis.

3. <u>Ideals Of Operators</u> : Though the summability techniques of Pietsch furnish a much easier tool to study nuclear spaces, the most elegant method effective for suitable generalization of the concepts of nuclearity and Schwartzarity is via the use of operator ideals as proposed by Pietsch [90]. The fact that nuclear spaces can be characterized by means of the ideal of nuclear operators in the same way as Schwartz spaces are characterized by means of the ideal of precompact operators, led Pietsch to introduce the concept of a locally convex A-space, related to an arbitrary operator ideal A, thus providing a general framework to study several classes of locally convex spaces. It turns out that most of the permanence properties of the class of locally convex A-spaces can be deduced from the general properties of the corresponding operator ideal A. Pietsch [90] has established under suitable assumptions on the ideal A, results concerning

completions, subspaces, quotient spaces, products and countable direct sums of locally convex spaces, whereas Apiola [3] has considered tensor products of locally convex A-spaces as well as spaces of linear operators. Concerning this theory for λ-nuclear and pseudo-λ-nuclear spaces defined respectively by the semi-ideals of λ-nuclear and pseudo-λ-nuclear operators, no attempt was made until 1977 when Apiola [4] employed the ideal-theoretic arguments to investigate the permanence properties of $\Lambda(P)$-and $\Lambda(Q)$-nuclear spaces with respect to tensor products and spaces of linear operators, $\Lambda(P)$ and $\Lambda(Q)$ being nuclear smooth sequence spaces of infinite and finite type, respectively. This work was further continued by the same author in [5] where these permanence properties were completely characterized in the case of $\Lambda_\infty(\alpha)$-, $\Lambda_1(\alpha)$- and $\Lambda_{\mathbb{N}}(\alpha)$-nuclearities in terms of the stability of the exponent α. In a latest contribution to operator-ideal theory of λ-nuclear spaces, Nelimarkka [80] has proved that the class of $\Lambda(P;\mathbb{N})$-nuclear spaces is not generated by an ideal, whenever $\Lambda(P)$ is not a power series space of infinite type. He also proved a similar result for $\Lambda_{\mathbb{N}}(\alpha)$-nuclear spaces under some additional assumption on α - thus solving a problem of Jarchow [48]. In yet another paper [81], he investigated the class of locally convex spaces generated by the ideal of approximable operators and solved partially a problem raised earlier by Ramanujan [92]. The ideal-theoretic

arguments have been further utilized by Nelimarkka [82] to investigate the Λ-space character of the spaces of holomorphic functions and holomorphic germs on locally convex Λ-spaces- thus unifying the corresponding results on nuclearity and Schwartz property of these spaces proved earlier by Boland [13], [14]; Waelbroeck [121] and several other authors. For a detailed account related to applications of operator-ideal-theoretic techniques to λ-nuclearity, we refer to an interesting work of Nelimarkka [79] which also contains a brief introduction to the theory of locally convex A-spaces defined by an operator ideal A. Last, not the least, we refer to the monumental work of A. Pietsch [90] for an exhaustive and comprehensive treatment of this subject matter on operator ideals.

We have briefly touched upon various aspects of the theory of nuclear spaces, its generalizations and a historical survey of its subsequent development in the past two decades. During this short span of time, this theory has developed considerably and evolved as one of the most fascinating branches of functional analysis, through the works of many exponents of this field. Needless to say, much of the motivation for carrying out the investigations embodied in the present work, has come from several of these contributions on nuclear and $\Lambda(P)$-nuclear spaces and operators. This has culminated into bringing forth many generalizations of these results in the present setting

of $\hat{\lambda}$-nuclearity associated with the sequence space $\lambda = \Lambda(P;\phi)$, where P is a nuclear Köthe set of increasing type and ϕ a suitable function on $\mathbb{R}^+$. A detailed investigation of this type of nuclearity leads to several analytical characterisations of $\hat{\Lambda}(P;\phi)$-nuclear spaces in terms of Schauder bases, Kolmogorov diameters as well as the diametral dimension. These results also include the 'kernel theorem' on $\hat{\Lambda}(P;\phi)$-nuclearity of spaces. An application of some of these results yields new conditions for the $\hat{\Lambda}(P;\phi)$-nuclearity of smooth sequence spaces and the spaces of continuous linear operators.

On the other hand, our work on λ-bases in locally convex spaces is greatly influenced by the works of several authors on Schauder bases in nuclear spaces. Besides laying down results on the interrelationship between several types of λ-bases and their impact on the structure of the space in question, we exploit the previous results to settle the natural questions motivated by those known from the theory of Schauder bases in nuclear spaces. The corresponding situation in the present context turns out to be pathological as our results are in marked constrast to those known from the classical theory.

Although the subject on λ-nuclearity has undergone tremendous development in the past few years, a lot has yet to be done as there are still many outstanding problems awaiting their solutions. One of these problems concerns the investigation of λ-nuclearity associated with a smooth sequence

space of finite type or an Orlicz sequence space. The question of characterising the λ-nuclearity of a space in terms of the metric order of its 0-neighbourhoods is yet another in the wide spectrum of unexplored problems concerning λ-nuclear spaces. The solution of these and many other unsolved problems will surely put the theory of λ-nuclear spaces on the threshold of a new era of development and mathematicians will continue to look forward to receiving glimpses of many new directions with utmost eagerness and curiosity!

CHAPTER 2

λ-NUCLEAR OPERATORS AND SPACES

1. Introduction : Ever since its inception, the theory of λ-nuclear spaces has occupied by and large the attention of a number of investigators in the past. In almost all cases, people have confined their attention to the study of λ-nuclear spaces by particularising the sequence space λ in question. In this direction, after Persson and Pietsch [84] and Brudovski [15], [16] considered the question of replacing ℓ^1 (in the definition of nuclear operators) by ℓ^p and s-the space of rapidly decreasing sequences - respectively, various generalizations of the concept of nuclearity have been considered by many authors, e.g. [66], [91], [93], [95], [98], [99], [100], [113] and [116]. For further details and historical remarks on λ-nuclear spaces and their ramifications, we refer back to Chapter 1, Part II and the survey article of Kamthan [53], where it has been observed that ''it is not an easy going task to dig out characterisations of λ-nuclear spaces unless we particularize the spaces in question''. This observation is borne out by the fact that the situation in characterising λ-nuclear spaces seems to be comparatively more pleasant than in the case of λ-nuclear maps. Accordingly, we study in this chapter the problem of characterising λ-nuclear spaces and establish several (possibly new)

characterisations of such spaces. Indeed this is achieved in Section 2 for the most general case when λ is taken to be an arbitrary sequence space. Further restrictions on λ, for instance, when λ is taken to be $\Lambda(P;\phi)$ where P is a nuclear Köthe set of increasing type and ϕ is a suitable function on $\mathbb{R}^+$, yield some interesting results on the characterisation of λ-nuclear spaces. The main result in this direction is Theorem 2.3.4 which is established in Section 3 and is used in a later chapter to determine the conditions for the $\hat{\Lambda}(P;\phi)$-nuclearity of spaces in certain concrete situations. The chapter concludes with the so-called 'kernel theorem' on $\hat{\Lambda}(P;\phi)$-nuclear spaces contained in the last section which also incorporates a result on the characterisation of a $\hat{\Lambda}(P;\phi)$-nuclear space in terms of its diametral dimension. Most of these results are motivated by their corresponding analogues already developed in the theory of nuclear spaces; and these results also envelop a number of those recently established in the theory of λ-nuclear spaces.

2. λ-<u>Nuclearity</u> : In this section we introduce the basic definitions of different types of 'λ-nuclear' maps and study the associated nuclearity of locally convex spaces.

<u>Definition</u> 2.2.1 : Let $T \in L(X,Y)$ and λ be an arbitrary sequence space. Then T is said to be

(i) λ-nuclear, if there exist $a \in \lambda$, an equicontinuous sequence $\{f_n\} \subset X^*$ and a sequence $\{y_n\} \subset Y$ with $\{\langle y_n, g\rangle\} \in \lambda^\times$ for each $g \in Y^*$, such that

$$T(x) = \sum_{n\geq 1} a_n \langle x, f_n\rangle\, y_n,\ x \in X$$

(ii) pseudo-λ-nuclear (or $\hat{\lambda}$-nuclear), provided T satisfies all the conditions laid down in (i) with $\lambda^\times$ being replaced by ℓ^∞, and

(iii) quasi-λ-nuclear, if for each $p \in \mathcal{D}_Y$, there exist $a \in \lambda$ and an equicontinuous sequence $\{f_n\} \subset X^*$ such that

$$p(Tx) \leq \sum_{n\geq 1} |a_n \langle x, f_n\rangle|,\ x \in X.$$

The collection of all such operators (in (iii)) will be denoted by $N_\lambda^q(X,Y)$.

Definition 2.2.2 : A seminorm p on an l.c. TVS X is called quasi-λ-nuclear, provided there exist $\{\zeta_n\} \in \lambda$ and an equicontinuous sequence $\{f_n\} \subset X^*$ such that

$$p(x) \leq \sum_{n\geq 1} |\zeta_n \langle x, f_n\rangle|,\quad x \in X.$$

We have the simple

Proposition 2.2.3(a) : The composition of any of the operators in Definition 2.2.1 with a continuous linear operator is of the same kind.

Proof : We prove the result for quasi-λ-nuclear operators. Thus, let $T \in L(X,Y)$ and $S \in N_\lambda^q(Y,Z)$, then for $p \in \mathcal{D}_Z$, we have

$$(*) \qquad p[(ST)(x)] \leq \sum_{n\geq 1} |\zeta_n \langle T(x), f_n \rangle|, \quad x \in X$$

where $\{\zeta_n\} \in \lambda$ and $\{f_n\} \subset Y^*$ is equicontinuous. If we write $g_n = T^* f_n$, $n \geq 1$, then the inequality

$$|\langle x, g_n \rangle| = |\langle x, T^* f_n \rangle| = |f_n(Tx)| \leq q'(Tx), \; n \geq 1, \quad x \in X$$

where $q' \in \mathcal{D}_Y$, yields that $\{g_n\} \subset X^*$ is equicontinuous because $q' \circ T \in \mathcal{D}_X$. Thus $(*)$ can be rewritten in the form

$$p[(ST)(x)] \leq \sum_{n\geq 1} |\zeta_n \langle x, g_n \rangle|, \quad x \in X$$

so that $ST \in N_\lambda^q(X,Z)$.

The proofs of the following three results can be disposed of similarly.

Proposition 2.2.3 (b) : For a normal sequence space λ, the composition of a continuous linear operator with an operator of λ-type yields an operator of the same type.

Proposition 2.2.3 (c) : An operator $T \in L(X,Y)$ belongs to $N_\lambda^q(X,Y)$ if and only if the seminorm $q \circ T$ is quasi-λ-nuclear on X, for each $q \in \mathcal{D}_Y$.

Proposition 2.2.3 (d) : An operator $T \in L(X,Y)$ is quasi-λ-nuclear if and only if there exists a quasi-λ-nuclear seminorm

$p \in \mathcal{D}_X$ such that the set $A_p = \{Tx : p(x) \leq 1\}$ is bounded in Y.

Note : It follows from Proposition 2.2.3 (a) that each class of operators in Definitions 2.2.1 forms a semi-ideal in the sense of Definition 0.6.13.

Definition 2.2.4 : An l.c. TVS X is said to be λ-nuclear (resp. $\hat{\lambda}$-nuclear) provided for each $u \in \mathcal{U}_X$, there exists $v \in \mathcal{U}_X$, $v < u$ such that the map $\hat{\phi}_u^v : \hat{X}_v \to \hat{X}_u$ is λ-nuclear (resp. $\hat{\lambda}$-nuclear).

The following theorem on the $\hat{\lambda}$-nuclearity of an l.c. TVS is the main result of this section.

Theorem 2.2.5 : For an l.c. TVS X, the following conditions are equivalent:

(i) X is $\hat{\lambda}$-nuclear.

(ii) For each $u \in \mathcal{U}_X$, the map $\hat{\phi}_u : X \to \hat{X}_u$ is $\hat{\lambda}$-nuclear.

(iii) Each continuous linear mapping from X into any Banach space F is $\hat{\lambda}$-nuclear.

Proof : (i) ==> (ii) follows from Proposition 2.2.3(a), since $\hat{\phi}_u = \hat{\phi}_u^v \circ \hat{\phi}_v$.

(ii) ==> (i) : Let $u \in \mathcal{U}_X$. By the given hypothesis, the map $\hat{\phi}_u : X \to \hat{X}_u$ is $\hat{\lambda}$-nuclear. Thus

$$(*) \qquad \hat{\phi}_u(x) = \sum_{n \geq 1} \alpha_n f_n(x)\, x_u^{(n)}, \; x \in X$$

with usual restrictions on $\{\alpha_n\}$, $\{f_n\} \subset X^*$ and $\{x_u^{(n)}\} \subset \hat{X}_u$. Let $v_1 = \{x \in X: |f_n(x)| \leq 1, \; \forall\, n \geq 1\}$. Since $\{f_n\}$ is equicontinuous, $v_1 \in \mathcal{U}_X$. Suppose $v = v_1 \cap u$. Then $v \in \mathcal{U}_X$ and $v \subset u$.

Define

$$\hat{f}_n : X_v \to \mathbb{K}$$

by

$$\hat{f}_n(x_v) = f_n(x), \; x_v \in X_v, \; x_v = x + \ker p_v.$$

Observe that $\hat{f}_n$ is well defined because

$$|f_n(x)| \leq p_{v_1}(x) \leq p_v(x), \quad \forall\, x \in X,$$

and since this inequality is also valid for each $n \geq 1$, we find that for each $x_v \in X_v$,

$$|\hat{f}_n(x_v)| \leq \hat{p}_v(x_v), \quad \forall\, n \geq 1,$$

thereby yielding the equicontinuity of $\{\hat{f}_n\} \subset (\hat{X}_v)^*$. Further, $v < u$, therefore we have the well-defined mapping $\phi_u^v : X_v \to X_u$. Hence, making use of (*) we get, for each $x_v \in X_v$,

$$x_u = \phi_u^v(x_v) = \sum_{n \geq 1} \alpha_n \hat{f}_n(x_v)\, x_u^{(n)},$$

where $\{\alpha_n\} \in \lambda$, $\{\hat{f}_n\} \subset (X_v)^*$ is equicontinuous and $\{x_u^{(n)}\} \subset \hat{X}_u$ is bounded. Thus $\hat{\phi}_u^v$ is $\hat{\lambda}$-nuclear. Consequently X is $\hat{\lambda}$-nuclear.

(ii) ==> (iii). Let $T: X \to F$ be a continuous linear operator. Thus there exists $p \in \mathcal{D}_X$ such that

$$||T(x)|| \leq p(x), \quad \forall\, x \in X.$$

Put $v = \{x \in X: p(x) \leq 1\}$. Then $v \in \mathcal{U}_X$ and, therefore, by the given hypothesis, $\hat{\phi}_v: X \to \hat{X}_v$ is $\hat{\lambda}$-nuclear.

Define

$$g: X_v \to F$$

by

$$g(x_v) = T(x), \quad x_v = x + \ker p_v .$$

Since

$$||T(x) - T(y)|| \leq p(x-y) = 0$$

for $x, y \in x_v$, the mapping g is well-defined. The continuity of g follows from the inequality

$$||g(x_v)|| = ||T(x)|| \leq p(x) = \hat{p}_v(x_v), \; x_v \in X_v.$$

Further

$$g \circ \phi_u(x) = g(x_v) = T(x)$$

and, therefore, T is $\hat{\lambda}$-nuclear by Proposition 2.2.3(a).

(iii) ==> (iv) follows trivially because $\hat{X}_u$ is a Banach space for each $u \in \mathcal{U}_X$.

<u>Note</u> : The proof of the foregoing theorem suggests that the following is also true, namely

<u>Theorem</u> 2.2.5' : The following statements for an l.c. TVS X are equivalent :

(i) X is λ-nuclear.

(ii) For each $u \in \mathcal{U}_X$, the map $\hat{\phi}_u: X \to \hat{X}_u$ is λ-nuclear.

(iii) For each Banach space F, each continuous linear operator from X into F is λ-nuclear.

For our later use in this chapter, we also prove

<u>Theorem</u> 2.2.6 : The following statements for an l.c. TVS X are equivalent:

(i) For each $u \in \mathcal{U}_X$, there exists $v \in \mathcal{U}_X$, $v < u$ such that $\hat{\phi}_u^v : \hat{X}_v \to \hat{X}_u$ is quasi-λ-nuclear.

(ii) For each $u \in \mathcal{U}_X$, $\hat{\phi}_u : X \to \hat{X}_u$ is quasi-λ-nuclear.

(iii) $L(X,F) \subset N_\lambda^q(X,F)$, for each normed linear space F.

(iv) $L_b(X,Y) \subset N_\lambda^q(X,Y)$, for each l.c. TVS Y.

(v) $L_t(X,Y) \subset N_\lambda^q(X,Y)$, for each l.c. TVS Y.

<u>Proof</u> : The equivalence of (i), (ii) and (iii) follows on the lines of proof of Theorem 2.2.5.

(iii) ==> (iv). Let $T \in L(X,Y)$; then for some $u \in \mathcal{U}_X$ and $B \in \mathcal{B}_Y$, $T(u) \subset B$. Since $p_B(T(x)) \leq p_u(x)$ for each x in X, the map

$$T_B^u : X_u \to Y(B),$$

where

$$T_B^u(x_u) = T(x), \; x_u = x + \ker p_u, \; x \in X,$$

is a well-defined continuous linear mapping. By (iii),

$\phi_u \in N^q_\lambda(X,X_u)$. Next, observe that $T = i_B \circ T^u_B \circ \phi_u$ where i_B is the inclusion $Y(B) \to Y$. Applying Proposition 2.2.3(a), we conclude that $T \in N^q_\lambda(X,Y)$.

(iv) ==> (v) is obvious.

(v) ==> (iii). Let $T \in L(X,F)$. Then $T \in L_t(X,Y_\sigma)$ where $Y_\sigma = (F,\sigma(F,F^*))$ so that by the given hypothesis, $T \in N^q_\lambda(X,Y_\sigma)$. Thus by virtue of Proposition 2.2.3(d), there exists a quasi-λ-nuclear seminorm $p \in \mathcal{D}_X$ such that $A_p = \{Tx:p(x) \leq 1\}$ is bounded in Y_σ. But then A_p is bounded in the norm topology of F (cf. also Theorem 0.2.6). Therefore, by the same proposition, $T \in N^q_\lambda(X,F)$. Thus $L(X,F) \subset N^q_\lambda(X,F)$ and the result is established.

3. $\hat{\Lambda}(P;\phi)$-Nuclearity : In order to be able to obtain analytical characterizations of λ-nuclear spaces, we need impose certain restrictions on the sequence space λ. This is accomplished in the present section by choosing λ to be the Köthe space $\Lambda(P;\phi)$ where P is assumed to be a nuclear Köthe set of increasing type in the sense that $\Lambda(P)$ is a nuclear space and that the elements of P are positive increasing sequences. These conditions on P will be assumed throughout the rest of this chapter. We start with a lemma which establishes equivalence among several kinds of operators on Hilbert spaces, associated with a sequence space λ where $\lambda = \Lambda(P;\phi)$.

<u>Lemma</u> 2.3.1 : Let H_1 and H_2 be Hilbert spaces and $T \in L(H_1,H_2)$. Then the following statements are equivalent:

(i) T is $\hat{\Lambda}(P;\phi)$-nuclear.

(ii) T is quasi-$\Lambda(P;\phi)$-nuclear.

(iii) T is of $\Lambda(P;\phi)$-type.

(iv) T^* is of $\Lambda(P;\phi)$-type.

(v) T^* is $\hat{\Lambda}(P;\phi)$-nuclear.

(vi) T^* is quasi- $\Lambda(P;\phi)$-nuclear.

<u>Proof</u> : (i) ==> (ii) is obvious .

(ii) ==> (iii). By (ii) we have

$$(*) \qquad ||T(x)|| \leq \sum_{n\geq 1} |\alpha_n \langle x,f_n\rangle|, \quad x \in H_1 ,$$

with usual restrictions on $\{\alpha_n\}$ and $\{f_n\}$. For $n \in \mathbb{N}$, we define the closed subspace M_n of H_1 by

$$M_n = \{x \in H_1 : \langle x,f_i\rangle = 0, \ i=1,2,\ldots,n\} .$$

Clearly, codim $M_n \leq n$. By the well known decomposition theorem for Hilbert spaces, we have $H = M_n \oplus M_n^{\perp}$, $M_n^{\perp}$ being the orthogonal complement of M_n in H_1. Since $H_1/M_n \cong M_n^{\perp}$, $\dim M_n^{\perp} \leq n$.

Let U denote as usual the closed unit ball in H_1. Using (*), for each $x \in U \cap M_n$, we have

$$||T(x)|| \leq \sum_{i\geq n+1} |\alpha_i| = \zeta_n \text{ , say.}$$

Hence

$$T(U \cap M_n) \subset \zeta_n V,$$

V being the closed unit ball in H_2. Taking polars and applying Lemma of [114], p. 65, we obtain

$$T^*(V^o) \subset \zeta_n U^o + M_n^{\perp}$$

$$\Longrightarrow \quad \delta_n(T^*(V^o), U^o) \leq \zeta_n, \; n \geq 1.$$

Since $\{\zeta_n\} \in \Lambda(P;\phi)$ (cf. [79], p. 32) and

$$\alpha_n(T) = \alpha_n(T^*) = \delta_n(T^*(V^o), U^o), \quad \forall\, n \geq 1,$$

by Proposition 0.6.9, we find that $\{\alpha_n(T)\} \in \Lambda(P;\phi)$, thus establishing (iii).

(iii) ==> (i). Since $\{\alpha_n(T)\} \in \Lambda(P;\phi) \subset \ell^1$, it follows that T is nuclear (see [89], p. 138), hence compact and, therefore, by Theorem 0.6.10, we have

$$T(x) = \sum_{n \geq 1} a_n \langle x, e_n \rangle h_n \;, \quad x \in H_1$$

where $\{e_n\}$ and $\{h_n\}$ are orthonormal sequences in H_1 and H_2, respectively. But by Theorem 0.6.11, $\alpha_n(T) = a_n$, $n \geq 1$ yielding therefore that T is $\hat{\Lambda}(P;\phi)$-nuclear. Thus the result is completely established.

An immediate consequence of the preceding lemma is contained in

<u>Proposition</u> 2.3.2 : For an l.c. TVS X and the sequence $\Lambda(P;\phi)$, the following statements are equivalent:

(i) X is $\hat{\lambda}$-nuclear.

(ii) For each $u \in \mathcal{U}_X$, there exists $v \in \mathcal{U}_X$, $v < u$ such that $\hat{\phi}_u^v \in N_\lambda^q(\hat{X}_v, \hat{X}_u)$.

(iii) For each $u \in \mathcal{U}_X$, $\hat{\phi}_u \in N_\lambda^q(X, \hat{X}_u)$.

(iv) $L(X,F) = N_\lambda^q(X,F)$, for each normed space F.

(v) $L_b(X,Y) = N_\lambda^q(X,Y)$, for each l.c. TVS Y.

(vi) $L_t(X,Y) = N_\lambda^q(X,Y)$, for each l.c. TVS Y.

Proof : The equivalence of the conditions (ii) through (vi) follows from Theorem 2.2.6. (The equality in the last three conditions follows as a consequence of the fact that the elements of $N_\lambda^q(X,Y)$, being quasi-nuclear-because $\Lambda(P;\phi) \subset \ell^1$ - are a fortiori precompact operators). Also (i)<==> (ii) follows from Lemma 2.3.1, for, under each of the conditions (i) and (ii), $\hat{X}_u$ is a Hilbert space for each $u \in \mathcal{U}_X$(cf. Proposition 0.5.4).

The foregoing result generalizes a similar result on nuclear spaces proved in [97], p. 97.

Remark : The condition (iv) in the foregoing result cannot be strengthened to the case when F is replaced by an arbitrary l.c. TVS Y, as is testified by the following example.

Example 2.3.3 : Let X be an infinite dimensional $\Lambda(P)$-nuclear space, $\Lambda(P)$ being a nuclear G_∞-space. The identity mapping $i:X \to X$ being continuous, is not quasi-$\Lambda(P)$-nuclear for, otherwise, i would have to be precompact and, therefore, X would be finite dimensional by Theorem 0.2.7, which is contrary to our hypothesis on X.

We are now ready to prove the crucial

<u>Theorem</u> 2.3.4 : The following statements for an l.c. TVS X are equivalent:

(i) X is $\hat{\Lambda}(P;\phi)$-nuclear.

(ii) For every $u \in \mathcal{U}_X$, there exist $\{\zeta_n\} \in \Lambda(P;\phi)$ and an equicontinuous sequence $\{f_n\} \subset X^*$ such that

$$(**) \qquad p_u(x) \leq \sum_{n\geq 1} |\zeta_n \langle x, f_n \rangle|, \quad \forall\, x \in X.$$

(iii) To every $u \in \mathcal{U}_X$, there corresponds $v \in \mathcal{U}_X$, $v < u$ such that $\{\delta_n(v,u)\} \in \Lambda(P;\phi)$.

<u>Proof</u> (i) ==> (ii) : By the hypothesis, to each $u \in \mathcal{U}_X$, there corresponds $v \in \mathcal{U}_X$, $v < u$ such that the map

$$\hat{\phi}^v_u : \hat{X}_v \to \hat{X}_u$$

is $\hat{\Lambda}(P;\phi)$-nuclear and, therefore, quasi-$\Lambda(P;\phi)$-nuclear. Thus there exist $\{\zeta_n\} \in \Lambda(P;\phi)$ and an equicontinuous sequence $\{\hat{f}_n\} \subset (\hat{X}_v)^* = (X_v)^* \cong X^*(u^o)$ such that

$$\hat{p}_u(\hat{\phi}^v_u(x_v)) \leq \sum_{n\geq 1} |\zeta_n \langle x_v, \hat{f}_n \rangle|, \quad x_v \in X_v$$

$$\Longrightarrow \quad p_u(x) \leq \sum_{n\geq 1} |\zeta_n \langle x, f_n \rangle|, \quad x \in X.$$

(ii) ==> (iii). For $u \in \mathcal{U}_X$, the space $\hat{X}_u$ can be considered a Hilbert space since X is a nuclear space by virtue of (ii) (see [89], Proposition 4.14). Also there exists $v \in \mathcal{U}_X$ such that $\{f_n\} \subset v^o$. Consider the map

$$\hat{\phi}_u^v : \hat{X}_v \to \hat{X}_u .$$

Then (**) implies that

$$\hat{p}_u \left[\hat{\phi}_u^v(x_v) \right] \leq \sum_{n \geq 1} |\zeta_n \langle x_v, \hat{f}_n \rangle|, \quad x_v \in \hat{X}_v$$

where $\{\hat{f}_n\} \subset (\hat{X}_v)^*$. Hence $\hat{\phi}_u^v$ is quasi-$\Lambda(P;\phi)$-nuclear and, therefore, by Lemma 2.3.1, $\hat{\phi}_u^v$ is of $\Lambda(P;\phi)$-type. Since $\delta_n(v,u) \leq \alpha_n(\phi_u^v) = \alpha_n(\hat{\phi}_u^v)$ by Proposition 0.6.9, we conclude that $\{\delta_n(v,u)\} \in \Lambda(P;\phi)$.

(iii) ==> (i). We have $\{\delta_n(v,u)\} \in \Lambda(P;\phi) \subset \ell^1$. Thus X is nuclear by [110], p. 15 and, therefore, the mapping $\hat{\phi}_u^v$ is nuclear and hence compact. Since $\hat{X}_u$ is a Hilbert space for each $u \in \mathcal{U}_X$, we have, by Proposition 0.6.9, $\alpha_n(\hat{\phi}_u^v) = \delta_n(v,u)$. Thus $\hat{\phi}_u^v$ is of $\Lambda(P;\phi)$-type and, therefore, $\hat{\Lambda}(P;\phi)$-nuclear by Lemma 2.3.1. This establishes the $\hat{\Lambda}(P;\phi)$-nuclearity of X.

<u>Remark</u> : The equivalence (i) <==> (ii) also follows from [79], Proposition 2.4.2 and the remark following it where $\Lambda(P;\phi)$ is replaced by any normal sequence space $\lambda \subset \ell^1$.

<u>Further restrictions on the power set P</u> : In this sub-section we establish certain results on the characterisation of $\hat{\Lambda}(P;\phi)$-nuclear spaces which are obtained by putting some additional constraints on P or ϕ. In the present situation we restrict P to be a <u>nuclear power set of infinite type</u>. We start with the following result where the G_∞-character of P yields an alternative criterion for the $\hat{\Lambda}(P;\phi)$-nuclearity

(of an l.c. TVS) which is slightly different from that given in the previous theorem.

<u>Proposition</u> 2.3.5 : An l.c. TVS X is $\hat{\Lambda}(P;\phi)$-nuclear if and only if for each $p \in \mathcal{D}_X$, there exist $\zeta \in \Lambda(P;\phi)$ and an equicontinuous sequence $\{f_n\} \subset X^*$ such that the inequality

$$p(x) \leq \sup_{n \geq 1} \{|\zeta_n \langle x, f_n \rangle|\} \quad ,$$

is valid for each $x \in X$.

<u>Proof</u> : The sufficiency part follows trivially from Theorem 2.3.4. To prove the necessity part, we again appeal to Theorem 2.3.4 to get, for each $p \in \mathcal{D}_X$, an equicontinuous sequence $\{f_n\} \subset X^*$ and $\{\lambda_n\} \in \Lambda(P;\phi)$ such that

$$p(x) \leq \sum_{n \geq 1} |\lambda_n \langle x, f_n \rangle| \quad , x \in X.$$

In virtue of the above inequality, it suffices to show that $\{n^k \lambda_n\} \in \Lambda(P;\phi)$ for each $k \geq 1$. Indeed, using Proposition 0.5.9, we get, corresponding to each $k \geq 1$, some $\rho \in P$ and $M > 0$ such that

$$n^k \leq M \rho_n, \quad \forall n \geq 1.$$

Now for $a \in P$, we have

$$\sum_{n \geq 1} \phi[n^k |\lambda_n|] a_n \leq \sum_{n \geq 1} n^k \phi(|\lambda_n|) a_n$$

$$\leq M \sum_{n \geq 1} \phi(|\lambda_n|) \rho_n a_n$$

$$\leq M \sum_{n \geq 1} \phi(|\lambda_n|) c_n < \infty ,$$

where $\rho_n a_n \leq c_n$, $n \geq 1$ for some $c \in P$, by the G_∞-character of P. Therefore, it follows that $\{2n^k \lambda_n\} \in \Lambda(P;\phi)$ and this completes the proof.

Before we prove our next result, we make the following definition.

<u>Definition</u> 2.3.6 : Let X be an l.c. TVS. A set $A \subset \omega$ is said to be <u>uniformly contained</u> in $\Delta(X)$–the diametral dimension of X–provided for each $u \in \mathcal{U}_X$, there exists $v \in \mathcal{U}_X$, $v < u$ such that $\lim_{n\to\infty} x_n \delta_n(v,u) = 0$, for each $x \in A$.

A characterisation of $\Lambda(P)$-nuclearity of an l.c. TVS in terms of its diametral dimension is contained in

<u>Proposition</u> 2.3.7 : An l.c. TVS X is $\Lambda(P)$-nuclear if and only if P is uniformly contained in $\Delta(X)$.

<u>Proof</u> : Suppose X is $\Lambda(P)$-nuclear. Let $u \in \mathcal{U}_X$. By Theorem 2.3.4, there exists $v \in \mathcal{U}_X$, $v < u$ such that $\{\delta_n(v,u)\} \in \Lambda(P)$, that is

$$\sum_{n\geq 1} \delta_n(v,u)\, a_n < \infty, \quad \forall\, a \in P.$$

In particular, we have $\lim_{n\to\infty} \delta_n(v,u)\, a_n = 0$, for each $a \in P$, yielding that P is uniformly contained in $\Delta(X)$.

Conversely, let P be uniformly contained in $\Delta(X)$. Thus for each $u \in \mathcal{U}_X$, there exists $v \in \mathcal{U}_X$, $v < u$ such that for each $a \in P$, there exists $M_a > 0$ satisfying

$$a_n \delta_n(v,u) \leq M_a, \quad \forall\, n \geq 1.$$

Choose $b \in P$ arbitrarily. By the nuclearity of $\Lambda(P)$, we get $c \in P$ with $\sum_{n\geq 1} 1/c_n < \infty$. Now the G_∞-character of P yields an element $a \in P$ such that $b_n c_n \leq a_n$, for $n \geq 1$. Therefore,

$$\delta_n(v,u)\, b_n \leq \delta_n(v,u)\, a_n/c_n \leq M_a/c_n, \ \forall\, n \geq 1$$

$$\implies \sum_{n\geq 1} \delta_n(v,u)\, b_n < \infty$$

$$\implies \{\delta_n(v,u)\} \in \Lambda(P).$$

Appealing to Theorem 2.3.4, we find that X is $\Lambda(P)$-nuclear.

<u>Definition</u> 2.3.8 : Let X and Y be l.c. TVS over the same field $\mathbb{K}$ and λ an arbitrary sequence space. A bilinear form $B: X \times Y \to \mathbb{K}$ is said to be λ-<u>nuclear</u> provided there exist $\zeta = \{\zeta_n\} \in \lambda$, an equicontinuous sequence $\{f_n\} \subset X^*$ and a sequence $\{g_n\} \subset Y^*$ satisfying $\{\langle y, g_n\rangle\} \in \lambda^\times$ for each $y \in Y$, such that B can be expressed as

$$B(x,y) = \sum_{n\geq 1} \zeta_n \langle x, f_n\rangle \langle y, g_n\rangle, \ \forall\, x \in X,\ y \in Y.$$

It is known ([89], p. 116) that an l.c. TVS X is nuclear if and only if for each Banach space F, all continuous bilinear forms on $X \times F$ are nuclear; it being understood that a nuclear bilinear form is nothing but a λ-nuclear bilinear form for $\lambda = \ell^1$. For a class of sequence spaces $\lambda \neq \ell^1$, we have the following characterisation of λ-nuclear spaces in terms of continuous bilinear forms. In the rest of this section we assume that ϕ satisfies the (additional) condition

(Δ) There exist $M(=M_\phi) \geq 1$ and $t_o \in \mathbb{R}^+$ such that $\phi(\sqrt{t}) \leq \sqrt{\phi(Mt)}$, $\forall\, t \in [0,t_o)$.

This condition yields the conclusion: $\Lambda(P;\phi) \subset \Lambda(P;\phi).\ \Lambda(P;\phi)$. Indeed, using Lemma 2.10 of [32], we find that $\{x_n^t\} \in \Lambda(P)$ if $\{x_n\} \in \Lambda(P)$. Thus if $x \in \Lambda(P;\phi)$, then $\{\phi(M|x_n|)\}$ and, therefore, $\{\sqrt{\phi(M|x_n|)}\}$ belongs to $\Lambda(P)$. Using the (Δ)-condition and the fact that $x_n \to 0$, we find that $\{\phi(\sqrt{|x_n|})\} \in \Lambda(P)$ so that $\{\sqrt{|x_n|}\} \in \Lambda(P;\phi)$ and, therefore, $\Lambda(P;\phi) \subset \Lambda(P;\phi).\ \Lambda(P;\phi)$.

We are now ready to prove the main result of this subsection in

<u>Theorem</u> 2.3.9 (<u>Kernel Theorem</u>) : An l.c. TVS X is $\Lambda(P;\phi)$-nuclear if and only if for each Banach space F, every continuous bilinear form on $X \times F$ is $\Lambda(P;\phi)$-nuclear.

<u>Proof</u> (<u>Necessity</u>) : This follows from [33], Theorem 2.13.

(<u>Sufficiency</u>) : Let $u \in \mathcal{U}_X$ be arbitrarily chosen. Consider the Banach space $X^*(u^o)$ and define a bilinear form

$$B: X \times X^*(u^o) \to \mathbb{K}$$

by

$$B(x,a) = \langle x,a \rangle, \quad x \in X, \quad a \in X^*(u^o).$$

Now

$$|B(x,a)| = |\langle x,a\rangle| = \left|\left\langle \frac{x}{p_u(x)}, \frac{a}{p_{u^o}(a)} \right\rangle\right| p_u(x) p_{u^o}(a)$$

$$\leq p_u(x) p_{u^o}(a), \ \forall\, x \in X,\ a \in X^*(u^o),$$

giving thereby the continuity of B. Hence, by the given hypothesis we have

$$(*) \qquad B(x,a) = \sum_{n\geq 1} \zeta_n \langle x,f_n\rangle\langle a,g_n\rangle, \; x \in X, \; a \in X^*(u^o),$$

where $\{\zeta_n\} \in \Lambda(P;\phi)$, $\{f_n\} \subset X^*$ is equicontinuous and $\{g_n\} \subset (X^*(u^o))^*$ with $\{\langle a,g_n\rangle\} \in [\Lambda(P;\phi)]^\times$ for each $a \in X^*(u^o)$. By the remark preceding this theorem, we have $\Lambda(P;\phi) \subset \Lambda(P;\phi).\Lambda(P;\phi)$. Thus there exist $\lambda, \eta \in \Lambda(P;\phi)$ with $\zeta_n = \lambda_n\eta_n$, $n \geq 1$ and, therefore, (*) yields the inequality

$$|\langle x,a\rangle| \leq \sum_{n\geq 1} |\lambda_n\langle x,f_n\rangle \; \eta_n\langle a,g_n\rangle|, \; x \in X, \; a \in X^*(u^o)$$

$$\leq \sum_{n\geq 1} |\lambda_n\langle x,f_n\rangle \; \eta_n p^*_{u^o}(g_n)|, \; \forall \; x \in X, \; a \in u^o$$

where

$$p^*_{u^o}(g_n) = \sup_{a\in u^o} |\langle a,g_n\rangle|.$$

Hence

$$\sup_{a\in u^o} |\langle x,a\rangle| \leq \sum_{n\geq 1} |\lambda_n\langle x,f_n\rangle \; \eta_n| p^*_{u^o}(g_n), \; \forall \; x \in X,$$

$$(**) \qquad \Longrightarrow p_u(x) \leq \sum_{n\geq 1} |\lambda_n\langle x,\hat{f}_n\rangle|, \quad \forall \; x \in X$$

with

$$\hat{f}_n = \eta_n p^*_{u^o}(g_n)\, f_n, \quad n \geq 1.$$

Now

$$\{\langle a,\eta_n g_n\rangle\} = \{\eta_n\langle a,g_n\rangle\} \in \Lambda(P;\phi).[\Lambda(P;\phi)]^\times \subset \ell^\infty ,$$

for each $a \in X^*(u^o)$; therefore, using the Banach-Steinhaus Theorem (cf. also Theorem 0.2.3), $\{\eta_n g_n\} \subset (X^*(u^o))^*$ is equicontinuous. Thus, using the fact that $\{f_n\}$ is equicontinuous, the inequality

$$|\hat{f}_n(x)| = p^*_{u^o}(\eta_n g_n)|f_n(x)| \leq M\ |f_n(x)| \leq M\ q(x),\ x \in X,$$

suggests that $\{\hat{f}_n\} \subset X^*$ is equicontinuous. Invoking Theorem 2.3.4, the (**) condition yields the $\hat{\Lambda}(P;\phi)$-nuclearity and therefore the $\Lambda(P;\phi)$-nuclearity of X.

λ-NUCLEARITY OF SOME CONCRETE SPACES

1. Introduction : This chapter is mainly devoted to the investigation of $\hat{\lambda}$-nuclearity of some spaces in certain concrete situations. Essentially we study in this chapter the $\hat{\Lambda}(P_0;\phi)$-nuclearity of Köthe sequence spaces and the spaces of linear operators. Here the results of the previous chapter are exploited to determine new conditions for the $\hat{\Lambda}(P_0;\phi)$-nuclearity of these locally convex spaces. In particular we prove the Grothendieck-Pietsch-Köthe type criterion in its most general form, which leads to the notion of uniform $\hat{\Lambda}(P_0;\phi)$-nuclearity of Köthe spaces. Apart from this, we also study the $\hat{\Lambda}(P_0;\phi)$-nuclearity of some sequence spaces of infinite and finite type.

In Section 2, we consider the space $L_\lambda(X,Y)$ of continuous linear operators endowed with the so-called 'λ-topology' which is obtained by gluing together the locally convex topologies of X and λ. Though there is a good deal of literature presently available on the Λ-nuclearity of $L(X,Y)$ when equipped with a topology of uniform convergence (see, for instance [4], [5]), nothing seems to be known on the Λ-nuclearity of $L_\lambda(X,Y)$ even when $\Lambda = \ell^1$. This is accomplished in Section 3 of this chapter where it is shown that the $\hat{\Lambda}$-nuclearity of these spaces is solely determined by the

behaviour of the sequence space λ which is used to define the 'λ-topology' on $L(X,Y)$. As in Chapter 2, P_0 will throughout stand for an arbitrary <u>nuclear Köthe set of increasing type</u>.

2. $\hat{\Lambda}(P_0;\phi)$-<u>Nuclearity of Köthe Spaces</u> : As has already been remarked in Chapter 2, it is not easy to characterize the $\hat{\lambda}$-nuclearity of an operator unless we particularize the sequence space λ or the spaces or the operator in question. However, it turns out that for diagonal operators acting between specific sequence spaces, the situation is entirely different. We recall that a diagonal operator $D: \lambda \to \mu$ defined by a sequence $\{d_n;\ n\geq 1\}$ of scalars is a mapping given by $D(x) = \{d_n x_n\}$, $x \in \lambda$. Below we prove that a $\hat{\lambda}$-nuclear diagonal map on ℓ^1 into itself (where $\lambda = \Lambda(P_0;\phi)$) can be completely characterized in terms of the diagonal elements d_n, $n\geq 1$. This result is then used to prove the Grothendieck-Pietsch-Köthe criterion for the $\hat{\Lambda}(P_0;\phi)$-nuclearity of Köthe spaces.

<u>Proposition</u> 3.2.1 : A diagonal map $D: \ell^1 \to \ell^1$ determined by a sequence $\{d_n;\ n\geq 1\}$, $(d_n \geq 0)$ is $\hat{\Lambda}(P_0;\phi)$-nuclear if and only if there exists an injection $\pi: \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N};\ d_n \neq 0\}$ such that $\{d_{\pi(n)}\} \in \Lambda(P_0;\phi)$.

<u>Proof</u> (<u>Necessity</u>) : Assume that $D(x) = \sum_{n\geq 1} \lambda_n \langle x, a^{(n)}\rangle y^{(n)}$, $x \in \ell^1$, with usual restrictions on $\{\lambda_n\}$, $\{a^{(n)}\}$ and

$\{y^{(n)}\}$. Since $\Lambda(P_o;\phi) \subset \ell^1$, D is nuclear and hence compact. Thus it follows that $d_n \to 0$. Indeed, using a characterisation of compact sets in ℓ^1 (cf. [58], p. 108 and [65], p. 282), we find that

$$d_n = |d_n| = \sum_{i \geq n} |d_i e_i^n| \leq \sup_{y \in U} \sum_{i \geq n} |d_i y_i| \leq \sup_{y \in D(U)} \sum_{i \geq n} |y_i|$$

where U denotes the closed unit ball in ℓ^1 and, therefore,

$$\lim_{n\to\infty} d_n \leq \lim_{n\to\infty} \sup_{y \in D(U)} \sum_{i \geq n} |y_i| = 0.$$

Hence we can find an injection $\pi: \mathbb{N} \to \mathbb{N}$ such that $d_{\pi(1)} \geq d_{\pi(2)} \geq d_{\pi(3)} \geq \dots \geq d_{\pi(n)} \geq \cdots$.

Now for an arbitrary $T \in F_n(\ell^1, \ell^1)$, we have by virtue of Theorem 0.2.1,

$$T(x) = \sum_{i=1}^{n-1} \lambda_i \langle x, a^{(i)} \rangle \, y^{(i)}, \ x \in \ell^1$$

where $a^{(i)} \in \ell^\infty$, $y^{(i)} \in \ell^1$ with $|\lambda_i| \leq ||T||$, $||a^{(i)}|| \leq 1$, $||y^{(i)}|| \leq 1$, $1 \leq i \leq n-1$. Thus for any $x \in \ell^1$ with $||x|| \leq 1$, we have

$$(*) \quad ||D-T|| \geq ||D(x) - T(x)||$$
$$= ||\{d_k x_k - \sum_{i=1}^{n-1} \lambda_i \langle x, a^{(i)} \rangle \, y_k^{(i)}\}_k||.$$

Let $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$ be a solution of the system of equations

$$\sum_{k=1}^{n} a_{\pi(k)}^{(i)} \zeta_k = 0, \ 1 \leq i \leq n-1,$$

satisfying

$$\sum_{k=1}^{n}{}' |\zeta_i| = 1.$$

If we take $x = \{x_i\}$ to be a sequence defined by

$$x_i = \begin{cases} \zeta_i \ , \text{ if } i = \pi(j),\ 1 \le j \le n \\ 0 \ , \text{ otherwise} \end{cases},$$

then $x \in U$ and

$$\langle x, a^{(i)} \rangle = \sum_{k=1}^{\infty} a_k^{(i)} x_k = \sum_{k=1}^{n} a_{\pi(k)}^{(i)} \zeta_k = 0,\ i \le 1 \le n-1.$$

Consequently, from (*) we obtain

$$\|D-T\| \ge \|\{d_k x_k\}\| = \sum_{k \ge 1} |d_k x_k| = \sum_{k=1}^{n} |d_{\pi(k)} \zeta_k|$$

$$\ge d_{\pi(n)} \sum_{k=1}^{n} |\zeta_k| = d_{\pi(n)},\ n=1,2,\ldots$$

and since this inequality is true for any $T \in F_n(\ell^1, \ell^1)$, we get

$$\alpha_n(D) = \inf_{T \in F_n(\ell^1, \ell^1)} \|D-T\| \ge d_{\pi(n)},\ \forall\ n = 1,2,\ldots\ .$$

By the normality of $\Lambda(P_o;\phi)$, it follows that $\{d_{\pi(n)}\} \in \Lambda(P_o;\phi)$ because $\{\alpha_n(D)\} \in \Lambda(P_o;\phi)$ by [79], p. 32.

<u>Sufficiency</u> : Suppose that $\{d_{\pi(n)}\} \in \Lambda(P_o;\phi)$ for some injection $\pi: \mathbb{N} \to \mathbb{N}$. If we set

$$M_d = \{n \in \mathbb{N} : d_n \ne 0\},$$

then we have

$$D(x) = \sum_{n\geq 1} \langle Dx, e^n\rangle e^n = \sum_{n\geq 1} d_n x_n e^n$$

$$= \sum_{n\in M_d} d_n x_n e^n = \sum_{n\in M_d} \langle Dx, e^n\rangle\, e^n$$

$$= \sum_{n\in M_d} \langle Dx, e^{\pi(n)}\rangle\, e^{\pi(n)} = \sum_{n\geq 1} d_{\pi(n)} \langle x, e^{\pi(n)}\rangle\, e^{\pi(n)}$$

where we have used the fact that $\{e^n; e^n\}$ is an unconditional base for ℓ^1. Since $\{d_{\pi(n)}\} \in \Lambda(P_0;\phi)$ and $||e^{\pi(n)}|| = 1$ for $n \geq 1$, it follows that D is $\hat{\Lambda}(P_0;\phi)$-nuclear.

The Grothendieck-Pietsch-Köthe criterion for the $\hat{\Lambda}(P_0;\phi)$-nuclearity of Köthe sequence spaces is contained in

Theorem 3.2.2 : A Köthe space $\Lambda(P)$ is $\hat{\Lambda}(P_0;\phi)$-nuclear if and only if to every $a \in P$, there correspond $b \in P$ and an injection $\pi: \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N} : a_n \neq 0\}$ such that

$$\left\{\frac{a_{\pi(n)}}{b_{\pi(n)}}\right\} \in \Lambda(P_0;\phi)\ .$$

Proof : By definition, $\Lambda(P)$ is $\hat{\Lambda}(P_0;\phi)$-nuclear if and only if for every $a \in P$, there exists $b \in P$ with $b \geq a$ such that the canonical mapping

$$\hat{\phi}_a^b : \hat{\Lambda}_b \to \hat{\Lambda}_a$$

is $\hat{\Lambda}(P_0;\phi)$-nuclear. Here

$$\Lambda_b = \frac{\Lambda(P)}{p_b^{-1}(0)}, \quad \Lambda_a = \frac{\Lambda(P)}{p_a^{-1}(0)}$$

and $\hat{\Lambda}_b$ and $\hat{\Lambda}_a$ denote their completions.

Let

$$M_a = \{n \in \mathbb{N} : a_n \neq 0\}.$$

It is well known that $\hat{\Lambda}_a$ and ℓ_a^1 are isomorphically identified via the isomorphism $\hat{\Psi}_a$ which is the unique continuous extension of Ψ_a, where $\Psi_a : \Lambda_a \to \ell_a^1$ is given by

$$\Psi_a(\hat{x}) = \{a_n x_n\}, \; \hat{x} \in \Lambda_a, \; \hat{x} = x + \ker p_a,$$

and

$$\ell_a^1 = \{x \in \ell^1 : x_n = 0, \; \forall\, n \notin M_a\}.$$

Put $\hat{D}_a^b = \hat{\Psi}_a \circ \hat{\phi}_a^b \circ \hat{\Psi}_b^{-1}$. Then $\hat{D}_a^b : \ell_b^1 \to \ell_a^1$ is a diagonal transformation determined by the sequence $\{\frac{a_n}{b_n}\}$. Now $\hat{\phi}_a^b$ is $\hat{\Lambda}(P_o;\phi)$-nuclear if and only if $\hat{D}_a^b$ is $\hat{\Lambda}(P_o;\phi)$-nuclear. But this is equivalent to the assertion that the associated diagonal map $D_a^b : \ell^1 \to \ell^1$ determined by the same sequence $\{a_n/b_n\}$, is $\hat{\Lambda}(P_o;\phi)$-nuclear. Applying the foregoing result, we find that there exists an injection $\pi : \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N} : \frac{a_n}{b_n} \neq 0\}$, such that

$$\left\{\frac{a_{\pi(n)}}{b_{\pi(n)}}\right\} \in \Lambda(P_o;\phi).$$

Since $n \in \pi(\mathbb{N})$ if and only if $a_n/b_n \neq 0$, the result is completely established.

The above result motivates the following definition:

<u>Definition</u> 3.2.3 : A Köthe space $\Lambda(P)$ is said to be <u>uniformly</u> $\hat{\Lambda}(P_0;\phi)$-nuclear provided there exists a (universal) permutation $\pi \in P(\mathbb{N})$ such that for each $a \in P$, there exists $b \in P$, $b \geq a$ satisfying $\{\frac{a_{\pi(n)}}{b_{\pi(n)}}\} \in \Lambda(P_0;\phi)$.

It can be easily verified (see Theorem 3.2.4, below) that uniform $\hat{\Lambda}(P_0;\phi)$-nuclearity of a Köthe space implies its $\hat{\Lambda}(P_0;\phi)$-nuclearity. It is proved in [96] that for a stable nuclear G_∞-space $\Lambda(P_0)$, these two notions are equivalent for metrizable Köthe spaces $\Lambda(P)$, provided ϕ is the identity function on $\mathbb{R}^+$. Below we prove that these two notions are the same in the case of smooth sequence spaces of either type.

<u>Theorem</u> 3.2.4 : The following conditions on a G_∞-space $\Lambda(P)$ are equivalent:

(i) $\Lambda(P)$ is $\hat{\Lambda}(P_0;\phi)$-nuclear.

(ii) For every $a \in P$, there exists $c \in P$, $c \geq a$ such that $\{1/c_n\} \in \Lambda(P_0;\phi)$.

(iii) There exists $a \in P$ with $\{1/a_n\} \in \Lambda(P_0;\phi)$.

(iv) $\Lambda(P)$ is uniformly $\hat{\Lambda}(P_0;\phi)$-nuclear.

<u>Proof</u> (i) ==> (ii). Let $a \in P$. By Theorem 2.3.4, there exists $c \in P$, $c \geq a$ such that $\{\delta_n(u_c,u_a)\} \in \Lambda(P_0;\phi)$. Let

$$L_n = \mathrm{sp}\ \{e_1,e_2,\ldots,e_n\}.$$

For $x \in L_n$, we have

$$p_c(x) = \sum_{i\geq 1} |x_i| c_i = \sum_{i=1}^{n} |x_i| c_i = \sum_{i=1}^{n} |x_i| a_i \frac{c_i}{a_i}$$

$$\leq \frac{c_n}{a_1} \sum_{i\geq 1} |x_i| a_i = \frac{c_n}{a_1} p_a(x).$$

Therefore, for $x \in u_a \cap L_n$, we obtain

$$p_c\left(\frac{a_1}{c_n} x\right) = \frac{a_1}{c_n} p_c(x) \leq 1$$

$$\Longrightarrow \quad \frac{a_1}{c_n}(u_a \cap L_n) \subset u_c .$$

By Theorem 0.6.12, we get $\delta_n(u_c, u_a) \geq \dfrac{a_1}{c_n}$, $\forall\, n \geq 1$.

$$\Longrightarrow \quad \left\{ \frac{1}{c_n} \right\} \in \Lambda(P_0; \phi).$$

(ii) ==> (iii). This is obvious.

(iii) ==> (iv). Let $a \in P$. There exists $c \in P$ with $\{1/c_n\} \in \Lambda(P_0;\phi)$. Also, there exists $d \in P$ such that $d \geq a$, $d \geq c$ and, therefore, $\{1/d_n\} \in \Lambda(P_0;\phi)$. For $a \in P$, there exists $b \in P$ with $a_n^2 \leq b_n$, $n \geq 1$. This leads to the assertion that $\{\frac{a_n}{b_n}\} \in \Lambda(P_0;\phi)$ so that $\Lambda(P)$ is uniformly $\hat{\Lambda}(P_0;\phi)$-nuclear.

(iv) ==> (i). The given hypothesis suggests the existence of a permutation $\pi_1 \in \mathcal{P}(\mathbb{N})$ such that to each $a \in P$, there corresponds $b \in P$ satisfying

$$(*) \qquad \left\{\frac{a_{\pi_1(n)}}{b_{\pi_1(n)}}\right\} \in \Lambda(P_0;\phi).$$

The implication follows trivially if $a \in sp\{e^n;\ n\geq 1\}$. Suppose $a \notin sp\{e^n;\ n\geq 1\}$. In this case the set $M = \{n \in \mathbb{N};\ a_n \neq 0\}$ is infinite. Thus we can arrange the set $\pi_1^{-1}(M)$ as an increasing sequence $\{n_1, n_2, \ldots, n_k, \ldots\}$. We now define the injection $\sigma : \mathbb{N} \to \mathbb{N}$ by $\sigma(k) = n_k$, $k \in \mathbb{N}$, so that $\sigma(\mathbb{N}) = \pi_1^{-1}(M)$ and σ is an increasing map. Also $a_n \neq 0 \iff n \in M \iff \pi_1^{-1}(n) = n_m$ for some $m \in \mathbb{N} \iff \pi_1^{-1}(n) = \sigma(m) \iff n = \pi_1\sigma(m) \iff n \in \pi_1\sigma(\mathbb{N})$. Therefore, if we define the injection $\pi : \mathbb{N} \to \mathbb{N}$ by $\pi(n) = \pi_1\sigma(n)$, we obtain $\pi(\mathbb{N}) = \{n \in \mathbb{N} : a_n \neq 0\}$. In view of Theorem 3.2.2, the result is completely established if we show that $\{\frac{a_{\pi(n)}}{b_{\pi(n)}}\} \in \Lambda(P_0;\phi)$; and this follows from the fact that σ is increasing and that the inequality

$$\sum_{n\geq 1} \frac{a_{\pi(n)}}{b_{\pi(n)}} c_n \leq \sum_{n\geq 1} \frac{a_{\pi_1\sigma(n)}}{b_{\pi_1\sigma(n)}} c_{\sigma(n)} \leq \sum_{n\geq 1} \frac{a_{\pi_1(n)}}{b_{\pi_1(n)}} c_n < \infty ,$$

is valid for each $c \in P_0$.

<u>Theorem</u> 3.2.5 : The following statements for a G_1-space $\Lambda(Q)$ are equivalent:

(i) $\Lambda(Q)$ is $\hat{\Lambda}(P_0;\phi)$-nuclear.

(ii) $Q \subset \Lambda(P_0;\phi)$.

(iii) $\Lambda(Q)$ is uniformly $\hat{\Lambda}(P_0;\phi)$-nuclear.

<u>Proof</u> (i) $\Rightarrow$ (ii) : Let $a \in Q$. By Theorem 2.3.4, we have $\{\delta_n(u_b, u_a)\} \in \Lambda(P_0;\phi)$ for some $b \in Q$, $b \geq a$. As

in the previous theorem, we get, for each $x \in L_n =$ sp $\{e_1, e_2, \ldots, e_n\}$, the inequality

$$p_b(x) \leq \frac{b_1}{a_n} p_a(x)$$

which leads to the inclusion

$$\frac{a_n}{b_1} (u_a \cap L_n) \subset u_b .$$

Using Theorem 0.6.12, we obtain

$$\delta_n(u_b, u_a) \geq \frac{a_n}{b_1} , \forall n \geq 1$$

$$\implies \quad a \in \Lambda(P_o;\phi)$$

$$\implies \quad Q \subset \Lambda(P_o;\phi).$$

(ii) ==> (iii). Let $a \in Q$. There exists $b \in Q$ with $a_n \leq b_n^2$, $n \geq 1$. Since $b \in Q \subset \Lambda(P_o;\phi)$, we get $\{\frac{a_n}{b_n}\} \in \Lambda(P_o;\phi)$ so that $\Lambda(Q)$ is uniformly $\hat{\Lambda}(P_o;\phi)$-nuclear.

(iii) ==> (i). This is proved in the preceding theorem.

3. Spaces Of Linear Operators : In this section we investigate the $\hat{\Lambda}(P_o;\phi)$-nuclearity of the space $L(X,Y)$ of continuous linear operators when it is equipped with a 'λ-topology' introduced earlier by De Grande-De Kimpe [23] and which besides other important cases, includes the well known 'ultrastrong' topology considered by J. von Neumann [107].

Given a locally convex space X and a sequence space λ, equipped with a normal topology as defined in Chapter 0 and which is compatible with $\langle\lambda,\lambda^{\times}\rangle$, we define the generalized

sequence space $\lambda\{X\}$ by

$$\lambda\{X\} = \{\bar{x} = \{x_n\}: x_n \in X, \{p_u(x_n)\} \in \lambda, \forall\, u \in \mathcal{U}_X\}.$$

The space $\lambda\{X\}$ will be assumed to carry with it the locally convex topology generated by the family $\{\pi_{s,u}: s \in \mathcal{S}, u \in \mathcal{U}_X\}$ of seminorms on $\lambda\{X\}$, where

$$\pi_{s,u}(\bar{x}) = p_s[\{p_u(x_n)\}], \text{ for } \bar{x} \in \lambda\{X\}.$$

We now define $L_\lambda(X,Y)$ to be the space $L(X,Y)$ equipped with the locally convex 'λ-<u>topology</u>' obtained by the generating family $\{\pi_{\bar{x},s,u}: \bar{x} \in \lambda\{X\}, s \in \mathcal{S}, u \in \mathcal{U}_Y\}$ of seminorms, where

$$\begin{aligned}\pi_{\bar{x},s,u}(T) &= \pi_{s,u}[\{T(x_n)\}] \\ &= p_s[\{p_u(Tx_n)\}],\ T \in L(X,Y).\end{aligned}$$

The following result on the topological dual of $L_\lambda(X,Y)$ under its λ-topology will be used in the proof of our main result (see [23], p. 172).

<u>Proposition</u> 3.3.1 : The topological dual of $L_\lambda(X,Y)$ can be identified with the space of all those linear functionals χ that can be written in the form

$$\chi(f) = \sum_{n\geq 1} \alpha_n \langle f(x_n), g_n\rangle, \quad f \in L(X,Y),$$

with $\{\alpha_n\} \in \lambda^\times$, $\{x_n\} \in \lambda\{X\}$ and the sequence $\{g_n\} \subset Y^*$ is equicontinuous.

It may be noted that the aforementioned result remains valid in the more general case when λ is equipped with a locally convex topology in which λ is an AK-space (cf. Proposition 0.4.4).

We are now ready to prove

<u>Theorem</u> 3.3.2 : For every sequence space $(\lambda, \mathcal{T}_S)$ which is $\hat{\Lambda}(P_o;\phi)$-nuclear, the space $L_\lambda(X,Y)$ is always $\hat{\Lambda}(P_o;\phi)$-nuclear.

<u>Proof</u>: Consider an arbitrary seminorm on $L_\lambda(X,Y)$ by fixing $\bar{x} \in \lambda\{X\}$, $u \in \mathcal{U}_Y$ and $s \in S$. Then using Theorem 2.3.4, we have

$$\pi_{\bar{x},s,u}(T) = p_s[\{p_u(Tx_n)\}]$$

$$\leq \sum_{i\geq 1} |\lambda_i < \{p_u(Tx_n)\}, F^{(i)}>|, \quad T \in L(X,Y)$$

where $\{\lambda_i\} \in \Lambda(P_o;\phi)$ and $\{F^{(i)}\} \subset \lambda^* = \lambda^\times$ is equicontinuous. By the Hahn-Banach theorem, we can select an equicontinuous sequence $\{f_n\} \subset Y^*$ such that $p_u(Tx_n) = |<Tx_n, f_n>|$, for $n \geq 1$. Therefore the foregoing inequality leads to

$$\pi_{\bar{x},s,u}(T) \leq \sum_{i\geq 1} |\lambda_i| (\sum_{n\geq 1} p_u(Tx_n) | F_n^{(i)}|)$$

$$= \sum_{i\geq 1} |\lambda_i| (\sum_{n\geq 1} |<Tx_n, f_n>| \, |F_n^{(i)}|)$$

$$= \sum_{i\geq 1} |\lambda_i| (| \sum_{n\geq 1} <Tx_n, f_n> g_n^{(i)} |),$$

where $g_n^{(i)} = \beta_n^{(i)} F_n^{(i)}$ with $|\beta_n^{(i)}| = 1$, for all $i, n \geq 1$. Since $\{g_n^{(i)}\} \in \lambda^{\times}$ for each $i \geq 1$, by Theorem 3.3.1 mentioned above, we find that the expression involving second term in the foregoing inequality represents a continuous linear functional, say $\hat{F}_i$ on $L_\lambda(X,Y)$, $i \geq 1$. In view of Theorem 2.3.4 we are done if we show that $\{\hat{F}_i\}$ is equicontinuous on $L_\lambda(X,Y)$ and this is achieved through the following argument:

The equicontinuity of $\{F^{(i)}\}$ on λ yields some $s \in \mathcal{S}$ so that for each $T \in L_\lambda(X,Y)$, we have

$$|\hat{F}_i(T)| = |\sum_{n\geq 1} F_n^{(i)} (\beta_n^{(i)} < Tx_n, f_n>)|$$

$$\leq p_s(\{\beta_n^{(i)}<Tx_n, f_n>\}_n)$$

$$= \sup_{\alpha\in s} \sum_{n\geq 1} |\alpha_n<Tx_n,f_n>|$$

$$= \sup_{\alpha\in s} \sum_{n\geq 1} |\alpha_n| p_u(Tx_n)$$

$$= \pi_{\bar{x},s,u}(T), \quad \text{for each } i \geq 1$$

and we are finished with the proof.

λ-BASES IN LOCALLY CONVEX SPACES

1. Introduction : The relationship of nuclearity of a locally convex space with its Schauder base is well known in the theory of nuclear spaces. Importance and further study of these spaces have led to their generalizations as λ-nuclear spaces. On the other hand, the significance of the absolute character of a Schauder base has been fully realized in view of its impact on the nuclear structure of a Fréchet space. Consequently, this notion of a base has become a subject of much interest and recent investigations, resulting in its further dissection and generalization to λ-bases [24]. This aforementioned development in the theory of locally convex spaces poses a natural question - motivated by the theory of Schauder bases in nuclear spaces - worth investigating, namely ''how λ-nuclearity of a space could be effectively related to the behaviour of a λ-base present there''. To be in a position to answer this question in as much generality as possible, there has to be available a well developed theory of λ-bases in locally convex spaces, necessary to have an insight into the behaviour and the impact of λ-bases on the structure of the underlying locally convex space. The earlier work in this direction has been carried out in [31] followed by that of De Grande-De Kimpe [24] where the notion

of a λ-base was first introduced and subsequently investigated.

In order to answer the question mentioned in the preceding lines, we start with an introduction to several notions of λ-bases in Section 2 and pay a good deal of attention to finding the basic properties of 'λ-bases' and their interrelationships. Section 3 deals with the results concerning the impact of λ-bases on the structure of the underlying space. We conclude this chapter with Section 4 where a special type of λ-bases, namely the Q-fully λ-bases have been singled out for further investigation. This section also incorporates several results on the characterisation of Q-fully λ-bases in Fréchet spaces. These results will be subsequently used in Chapter 5 to settle the question posed in the preceding paragraph.

2. λ-<u>Bases</u> : Before we embark upon a discussion of λ-bases in locally convex spaces, we need some preparation. Suppose $\{x_n; f_n\}$ is a Schauder base for an l.c. TVS X and λ is an arbitrary sequence space. Define

$$\Delta = \bigcap_{p \in \mathcal{D}_X} \frac{\lambda}{p^*}, \quad p^* \equiv \{ p(x_n) \}$$

and

$$\gamma = \Big\{ \{\alpha_i\} : \alpha_i \in \mathbb{K}, \{ \sum_{i=1}^{n} \alpha_i x_i \} \text{ is Cauchy in } X \Big\} .$$

Further, we say that a sequence space λ satisfies the (K)-<u>property</u> provided its Köthe dual $\lambda^\times$ contains an element $\beta^o \gg 0$ with $\inf_n \beta^o_n = k > 0$.

Remark : It follows trivially that every sequence space satisfying the (K)-property is contained in ℓ^1 and conversely. In particular every G_∞-space satisfies the (K)-property. Of course, there are non-perfect sequence spaces possessing the (K)-property; for instance, one may consider the spaces ℓ^p, $0 < p < 1$.

We are now in a position to introduce

Definition 4.2.1 : Let X be an l.c. TVS, $\{x_n; f_n\}$ a Schauder base for X and λ an arbitrary sequence space, equipped with its normal topology $\eta(\lambda,\lambda^\times)$. Then $\{x_n;f_n\}$ is said to be a

(i) semi-λ-base, if for each $p \in \mathcal{D}_X$, the mapping $\Psi_p: X \to \lambda$ is well defined where

$$\Psi_p(x) = \{p(x_n)\, f_n(x)\},\ x \in X.$$

(ii) pre-λ-base, if for each $p \in \mathcal{D}_X$, the mapping $\Phi_p: \gamma \to \lambda$ is well defined where

$$\Phi_p(\alpha) = \{\alpha_n\, p(x_n)\},\ \alpha \in \gamma .$$

(iii) λ-base, if $\Delta \subset \delta$ and for each $p \in \mathcal{D}_X$, the mapping $\Psi_p: X \to \lambda$ is well defined.

(iv) λ-pre-Köthe base, if it is a pre-λ-base and $\Psi_p: X \to \lambda$ is continuous.

(v) λ-Köthe base, if it is λ-pre Köthe and a λ-base, and

(vi) fully-λ-base, if it is a semi-λ-base and for each $p \in \mathcal{D}_X$, the mapping $\Psi_p : X \to \lambda$ is continuous.

It is easy to see that the above definitions can be reformulated in the form of

Definition 4.2.2 : With the same notation as in Definition 4.2.1 above, we say that $\{x_n; f_n\}$ satisfies

(i) $\iff \delta \subset \Delta$.

(ii) $\iff \gamma \subset \Delta$.

(iii) $\iff \delta = \Delta$.

(iv) $\iff \gamma \subset \Delta$ and the seminorm $Q_{p,y}$ is continuous on X; where for each $p \in \mathcal{D}_X$, $y \in \lambda^\times$,

$$Q_{p,y}(x) = \sum_{n\geq 1} p(x_n)\,|f_n(x)y_n|, \quad x \in X.$$

(v) $\iff \{x_n; f_n\}$ satisfies (iii) and (iv).

(vi) $\iff \{x_n; f_n\}$ satisfies (i) and $Q_{p,y}$ with $p \in \mathcal{D}_X$ and $y \in \lambda^\times$ is a continuous seminorm on X.

Definition 4.2.3 : A Schauder base $\{x_n; f_n\}$ in an l.c. TVS is said to be a Q-semi λ-base (resp. Q-pre λ-base, Q-λ-base, Q-λ-pre Köthe base, Q-λ-Köthe base, Q-fully λ-base) provided there exists a permutation $\pi \in \mathcal{P}(\mathbb{N})$ such that $\{x_{\pi(n)}; f_{\pi(n)}\}$ satisfies (i) (resp. (ii), (iii), (iv), (v) and (vi)) of Definition 4.2.1.

<u>Note</u> : In quite many statements of results which follow, the use of an arbitrary sequence space λ is frequent and implicit, although we do not mention it every time. If we restrict λ anyway, we shall specify the same explicitly.

A natural question that immediately arises as a result of the above definitions, is the independence and interrelationships, if any, of the foregoing several notions of 'λ-bases'. Concerning the latter, we have

<u>Proposition</u> 4.2.4 : Let X be an l.c. TVS possessing a Schauder base $\{x_n; f_n\}$; then we have the following implication diagram:

$$\begin{tikzcd}
(v) \arrow[r] \arrow[d] & (iv) \arrow[r] \arrow[d] & (vi) \arrow[d] \\
(iii) & (ii) \arrow[r] & (i)
\end{tikzcd}$$

The simple proof of these implications is omitted.

Concerning the reverse implications in the foregoing diagram, we have the following two results:

<u>Proposition</u> 4.2.5 : Let X be a sequentially complete l.c. TVS possessing a Schauder base $\{x_n; f_n\}$. Then

(a) (i) $\Longleftrightarrow$ (ii),

(b) (iv) $\Longleftrightarrow$ (vi),

and if, in addition, λ satisfies the (K)-property, then

(c) (i) $\Longleftrightarrow$ (iii).

Proof (a) : Let $\alpha \in \gamma$. Since X is sequentially complete, the series $\sum_{i\geq 1} \alpha_i x_i$ defines an element x in X. Thus $\alpha_n = f_n(x)$, $n \geq 1$ and, therefore, by (i), $\{\alpha_n\, p(x_n)\} \in \lambda$ so that $\Phi_p : \gamma \to \lambda$ is well defined for each $p \in \mathcal{D}_X$.

(b) The proof follows from (a).

(c) By Proposition 4.2.5, we have (iii) ==> (i). To complete the proof we show that $\Delta \subset \delta$. Thus if $\alpha \in \Delta$, then $\{\alpha_n\, p(x_n)\} \in \lambda$ for each $p \in \mathcal{D}_X$. Hence, using the (K)-property, we find that the series $\sum_{n\geq 1} \alpha_n x_n$ is absolutely convergent in X and, therefore, $x = \sum_{n\geq 1} \alpha_n x_n$ for some $x \in X$. Thus $\alpha = \{\alpha_n\} = \{f_n(x)\} \in \delta$.

Proposition 4.2.6 : Let X be a Mackey S-space possessing a Schauder base $\{x_n; f_n\}$. Then

(a) (i) <==> (vi).

(b) (ii) <==> (iv).

Proof (a) : In view of Proposition 4.2.4, it suffices to establish the implication (i) ==> (vi). Recall the sequence spaces δ and μ introduced in Chapter 0. Then we have

$$(*) \qquad \mu = \delta^{\beta} .$$

Indeed, the inclusion $\mu \subset \delta^{\beta}$ is straightforward in view of the fact that for each $f \in X^*$, the series $\sum_{n\geq 1} f(x_n)\, f_n(x)$ converges for each $x \in X$. For showing the reverse inclusion, let $\alpha \in \delta^{\beta}$. Since the nth sections $\alpha^{(n)}$ of α lie in μ

and $\alpha^{(n)} \to \alpha$ in $(\delta^\beta, \sigma(\delta^\beta, \delta))$, it follows that $\{\alpha^{(n)}\}$ is Cauchy in $(\mu, \sigma(\mu, \delta))$. Since $(X^*, \sigma(X^*, X))$ and $(\mu, \sigma(\mu, \delta))$ are topologically isomorphic, it follows from the sequential completeness of $(X^*, \sigma(X^*, X))$ that there exists an $\alpha^* \in \mu$ such that $\alpha^{(n)} \to \alpha^*$ in $(\mu, \sigma(\mu, \delta))$. Thus $\{\alpha_n\} = \{\alpha_n^*\}$, giving $\alpha \in \mu$ and, therefore, $\delta^\beta \subset \mu$. Hence (*) is verified.

Returning to the proof of (vi), it suffices to show that $\psi_p : X \to \lambda$ is $\sigma(X, X^*)$-$\sigma(\lambda, \lambda^\beta)$ continuous (see Proposition 0.2.8) because X is already a Mackey space and that $\eta(\lambda, \lambda^\times) \subset \tau(\lambda, \lambda^\times)$, by virtue of Theorem 0.2.5 and Proposition 0.3.3. Thus, let $y \in \lambda^\beta$ be arbitrarily chosen. By (i), the series $\sum_{i \geq 1} p(x_i)\, f_i(x)\, x_i$ converges for each $x \in X$. Hence $\{p(x_i)\, y_i\} \in \delta^\beta$ and using (*), there exists $f \in X^*$ such that $f(x_i) = p(x_i)\, y_i$, $i \geq 1$. Thus for each $x \in X$, we have

$$\left| \sum_{i \geq 1} p(x_i)\, f_i(x) y_i \right| = \left| \sum_{i \geq 1} f(x_i) f_i(x) \right| = |f(x)|,$$

establishing, therefore, the $\sigma(X, X^*)$- $\sigma(\lambda, \lambda^\times)$ continuity of ψ_p. This completes the proof of (a).

(b) Here once again, it is sufficient to prove that (ii) ==> (iv). Since (ii) ==> (i), the continuity of ψ_p can be proved as in (a), (i) ==> (ii) above. This leads to the implication (ii) ==> (iv) and the result is completely established.

Note : The line of proof of (i) ==> (vi) is almost similar to that of [24], p. 511; however, we have included its proof for the sake of completeness.

Remark: Concerning the independence of these several notions of λ-bases, we shall not linger on the same in the setting of the present more general case; however, when $\lambda = \ell^1$, one can find several examples in [56] and [57], distinguishing one type of an 'ℓ^1-base' from another.

We conclude this section with the following result which provides an analytical characterisation of fully-λ-bases when λ is assumed to be nuclear in its natural topology $\eta(\lambda,\lambda^\times)$.

Proposition 4.2.7 : Let $(\lambda, \eta(\lambda,\lambda^\times))$ be a perfect nuclear sequence space. Then a Schauder base $\{x_n; f_n\}$ in an l.c. TVS X is a fully-λ-base if and only if to each $p \in \mathcal{D}_X$ and $y \in \lambda^\times_+$, there corresponds $q \in \mathcal{D}_X$ such that

$$\sup_{n\geq 1} \{|f_n(x)|\ p(x_n)\ y_n\} \leq q(x) \quad , \ \forall\ x \in X.$$

Proof : We prove only the ''if'' part, the ''only if'' part being obvious. Let, therefore, $p \in \mathcal{D}_X$ and $y \in \lambda^\times_+$ be arbitrarily chosen. The nuclearity of $(\lambda, \eta(\lambda,\lambda^\times))$ yields some $z \in \lambda^\times_+$ such that $M = \sum_{n\geq 1} y_n/z_n < \infty$ (cf. Proposition 0.5.7). By the given hypothesis, we can find $q \in \mathcal{D}_X$ such that

$$\sup_{n\geq 1} \{ |f_n(x)| \, p(x_n) z_n \} \leq q(x), \quad x \in X.$$

Since the inequality

$$\sum_{n\geq 1} |f_n(x)| p(x_n) y_n \leq M \sup_{n\geq 1} \{ |f_n(x)| p(x_n) z_n \} \leq Mq(x)$$

is valid for each $x \in X$, the continuity of Ψ_p is verified and as λ is perfect, $\{ f_n(x) \, p(x_n) \} \in \lambda$. Hence $\{ x_n ; f_n \}$ is a fully-λ-base for X and the proof is completed.

Remark : It follows from the proof of the above proposition that if λ is an arbitrary sequence space, then a semi-λ-base $\{ x_n ; f_n \}$ in an l.c. TVS X is a fully-λ-base if and only if for each $p \in \mathcal{D}_X$ and $y \in \lambda^{\times}_{+}$, there exists $q \in \mathcal{D}_X$ such that

$$\sup_{n\geq 1} \{ |f_n(x)| \, p(x_n) y_n \} \leq q(x), \quad \forall \, x \in X.$$

3. Impact Of λ-Bases : The restriction of the (K)-property on a given sequence space λ yields several interesting applications of λ-bases; indeed, these applications reveal the impact which the λ-bases carry on the structure of the underlying locally convex space. Without further reference, it will be understood throughout the rest of this section that the sequence space occurring in any result, possesses the (K)-property. We now begin with

Proposition 4.3.1 : Let X be a sequentially complete

l.c. TVS having a fully-λ-base $\{x_n; f_n\}$. Then X can be (topologically) identified with a Köthe space $\Lambda(P_o)$, where the Köthe set P_o is given by

$$P_o = \{\{ p(x_n)y_n\} : p \in \mathcal{D}_X,\ y \in \lambda_+^{\times} \}.$$

<u>Proof</u> : In view of the existence of the mapping ψ_p, the function $\Psi : X \to \Lambda(P_o)$ where $\Psi(x) = \{ f_n(x) \}$, $x \in X$, defines a linear mapping. We proceed to show that Ψ is a topological isomorphism. It is clearly injective because $\{x_n; f_n\}$ is a Schauder base in X. To prove the surjectivity of Ψ, let $\alpha \in \Lambda(P_o)$. Then for $p \in \mathcal{D}_X$, we have

$$\sum_{n\geq 1} |\alpha_n|\ p(x_n) \leq \frac{1}{k} \sum_{n\geq 1} |\alpha_n|\ p(x_n)\ \beta_n^o < \infty.$$

Since X is sequentially complete, there exists $x \in X$ such that

$$x = \sum_{n\geq 1} \alpha_n x_n ,$$

giving $\Psi(x) = \{\alpha_n\}$. For proving the continuity of Ψ, let $\alpha \in P_o$. Then $\alpha_n = p(x_n)y_n$, $n \geq 1$, for some $p \in \mathcal{D}_X$ and $y \in \lambda_+^{\times}$. Thus we get

$$\hat{p}_\alpha(\Psi(x)) = \sum_{n\geq 1} p(x_n)|f_n(x)|y_n = Q_{p,y}(x) \leq q(x),\ x \in X,$$

by the continuity of $Q_{p,y}$. The continuity of Ψ^{-1} now follows from the inequality

$$p(\psi^{-1}(\alpha)) \leq \frac{1}{k} \sum_{n\geq 1} |\alpha_n| \beta_n^o \, p(x_n) = \frac{1}{k} \hat{p}_{\beta^o, p}(\alpha), \ \alpha \in \Lambda(P_o)$$

where $p \in \mathcal{D}_X$ and $\hat{p}_{\beta^o,p}$ denotes the seminorm on $\Lambda(P_o)$ resulting from the sequence $\{\beta_n^o \, p(x_n)\}$ in P_o.

Proposition 4.3.2 : Let $X \equiv (X, \mathcal{F})$ be an l.c. TVS having a fully-λ-base $\{x_n; f_n\}$. Then $\mathcal{F}$ and $\sigma(X, X^*)$ define the same Cauchy and convergent sequences in X.

Proof : The (K)-property of λ together with the fact that $\{x_n; f_n\}$ is a fully-λ-base in X suggests that the family $\{Q_{p,\beta^o} : p \in \mathcal{D}_{\mathcal{F}}\}$ of seminorms generates the topology on X equivalent to $\mathcal{F}$. Also, since for each $p \in \mathcal{D}_{\mathcal{F}}$, the mapping

$$\psi_p : X \to \lambda, \ \psi_p(x) = \{p(x_n) \, f_n(x)\}, \ x \in X,$$

is $\mathcal{F}$-$\eta(\lambda, \lambda^{\times})$ continuous, the adjoint map $\psi_p^* : \lambda^* \to X^*$ is well defined. In particular, $\psi_p^*(\zeta) \in X^*$ for each $\zeta \in \lambda^* = \lambda^{\times}$. Suppose now $z^n \to 0$ in $\sigma(X, X^*)$. Then

$$\langle z^n, \psi_p^*(\zeta) \rangle \to 0, \ \forall \ \zeta \in \lambda^{\times},$$

$$\Longrightarrow \langle \psi_p(z^n), \zeta \rangle \to 0, \ \forall \ \zeta \in \lambda^{\times}$$

$$\Longrightarrow \psi_p(z^n) \to 0 \ \text{in} \ \sigma(\lambda, \lambda^{\times}).$$

Hence by Proposition 0.3.1, $\psi_p(z^n) \to 0$ in $\eta(\lambda, \lambda^{\times})$. Therefore, $Q_{p,\beta^o}(z^n) \to 0$ for each $p \in \mathcal{D}_{\mathcal{F}}$. Thus $z^n \to 0$ in $\mathcal{F}$ and the proof is completed.

The following result concerns the structure of continuous linear functionals on an l.c. TVS possessing a fully-λ-base. More precisely, we have

Proposition 4.3.3 : Let X be an l.c. TVS equipped with a fully-λ-base $\{x_n; f_n\}$. Then the topological dual X^* of X consists precisely of those linear functionals f on X that can be written in the form

$$(*) \qquad f(x) = \sum_{n\geq 1} \alpha_n f_n(x), \quad x \in X$$

where for some $p \in \mathcal{D}$, the sequence $\{\frac{\alpha_n}{p(x_n)}\} \in \lambda^\times$.

Proof : Suppose f is a linear functional on X for which $(*)$ is true. Then

$$|f(x)| \leq \sum_{n\geq 1} |\alpha_n f_n(x)|, \ \forall \ x \in X.$$

As $\{\frac{\alpha_n}{p(x_n)}\} \in \lambda^\times$, there exists some $\beta \in \lambda^\times$ such that

$$|f(x)| \leq \sum_{n\geq 1} |f_n(x) \ \beta_n| \ p(x_n),$$

i.e., $|f(x)| \leq Q_{p,\beta}(x)$, $x \in X$ and, therefore, $f \in X^*$.

Conversely, suppose that $f \in X^*$. There exist $p \in \mathcal{D}$ and $K > 1$ such that

$$|f(x_n)| \leq K \ p(x_n), \quad \forall \ n \geq 1.$$

Now

$$x = \sum_{n\geq 1} f_n(x) \ x_n, \ x \in X,$$

and, therefore,

$$f(x) = \sum_{n\geq 1} f_n(x)\, f(x_n) = \sum_{n\geq 1} \alpha_n f_n(x)\,,$$

where $\alpha_n = f(x_n)$, $n \geq 1$, say. We are done if we show that $\{\frac{\alpha_n}{p(x_n)}\} \in \lambda^\times$ and this follows from the inequality

$$\sum_{n\geq 1} \left|\beta_n \frac{\alpha_n}{p(x_n)}\right| = \sum_{n\geq 1} \left|\beta_n \frac{f(x_n)}{p(x_n)}\right| \leq K \sum_{n\geq 1} |\beta_n| < \infty\,,$$

where $\beta \in \lambda$ is arbitrary.

The foregoing results evelope those proved in [56] where it is also shown that a barrelled space having an absolute base is complete. The following result shows that a more general form of this statement is valid (under somewhat mild restrictions) in case of λ-bases when λ is considered to be different from the Lebesgue space ℓ^1.

Proposition 4.3.4 : Let λ be a perfect sequence space and X a Mackey S-space containing a λ-base. Then X is complete.

Proof : Let $\{x_n; f_n\}$ denote the given λ-base in X. First of all we observe that under the given conditions, δ is perfect. Indeed, let $z \in \delta^{\times\times}$, then

$$\sum_{n\geq 1} |z_n u_n| < \infty, \ \forall\, u \in \delta^\times\,.$$

Since $\delta = \Delta$, we have

$$\sum_{n\geq 1} |f_n(x) y_n| p(x_n) < \infty\,, \ \forall\, x \in X,\ y \in \lambda^\times \text{ and } p \in \mathcal{D}_X,$$

yielding that $\{y_n p(x_n)\} \in \delta^{\times}$, for each $y \in \lambda^{\times}$ and $p \in \mathcal{D}_X$. Hence

$$\sum_{n\geq 1} |y_n z_n| p(x_n) < \infty, \ \forall\ y \in \lambda^{\times} \text{ and } p \in \mathcal{D}_X.$$

Therefore, $\{z_n p(x_n)\} \in \lambda^{\times\times} = \lambda$ for each $p \in \mathcal{D}_X$. Thus $z \in \Delta = \delta$ so that δ is perfect. Further, we have $\mu = \delta^{\beta} = \delta^{\times}$ (cf. proof of Proposition 4.2.6(a)). Hence $(X, \sigma(X, X^*))$, being topologically isomorphic with $(\delta, \sigma(\delta, \delta^{\times}))$, is sequentially complete because the latter is sequentially complete by Proposition 0.3.2. Since $\{x_n; f_n\}$ is a <u>fortiori</u> a semi-λ-base, it is also a fully-λ-base by Proposition 4.2.6(a). Thus it follows from Proposition 4.3.1 that X can be identified with a Köthe space $\Lambda(P_o)$ which is complete in its Köthe topology so that X itself is complete (the final argument for proving completeness of X may also be given as follows: in fact, by now we have shown that X is a Mackey S-space having a $\sigma(X, X^*)$-Schauder base such that X is $\sigma(X, X^*)$ - sequentially complete; and now apply Corollary 2.3 of [55] to get the completeness of X).

$\Lambda(P)$-<u>Bases</u> : In the subsequent sections we shall have occasion to make use of '$\Lambda(P)$-bases' and some other results depending upon it proved earlier when $\Lambda(P)$ is a Köthe sequence space. However, in such results we consider the Köthe topology $\mathcal{T}_P$ on $\Lambda(P)$ rather than its normal topology. In general, the results on $\Lambda(P)$-bases involving the topology

$\mathcal{F}_P$ do not follow from the corresponding results on $\Lambda(P)$-bases when $\Lambda(P)$ is considered to be equipped with its normal topology and vice versa. But in any case, the methods of proof in these two sets of results follow more or less on similar lines and argument. As an illustration, we mention the following results without proof. In the rest of this section, we write Λ for a fixed Köthe sequence space $\Lambda(P)$ equipped with its Köthe topology $\mathcal{F}_P$.

Proposition 4.3.5 : Let X be a sequentially complete l.c. TVS possessing a fully-Λ-base where $\Lambda \subset \ell^1$. Then X can be topologically identified with a Köthe space $\Lambda(P_o)$ where

$$P_o = \{ \{p(x_n)a_n\} : p \in \mathcal{D}_X, \ a \in P\} .$$

Proposition 4.3.6 : Let Λ be nuclear. Then a Schauder base $\{x_n; f_n\}$ in an l.c. TVS X is a fully-Λ-base if and only if for each $p \in \mathcal{D}_X$ and $a \in P$, there exists $q \in \mathcal{D}_X$ such that

$$\sup_{n \geq 1} \{ |f_n(x)| \, p(x_n)a_n\} \leq q(x), \ \forall \ x \in X.$$

4. Characterisation Of Q-fully λ-Bases : Having discussed in length the elementary properties of different types of λ-bases in the preceding sections, we now single out Q-fully λ-bases for our special attention. Indeed, this is the

class of 'λ-bases' which would finally be related with the λ-nuclearity of a space in question. The desired relationship is attained through the application of several results on the characterisation of Q-fully λ-bases to be established in the present section. The main result in this context is Theorem 4.4.4 where a Q-fully λ-base in a Fréchet space is characterized in terms of its dual basis. Throughout this section, we take λ to be a <u>perfect sequence space, unless specifically qualified, and equipped with its normal topology</u> $\eta(\lambda,\lambda^\times)$ <u>such that</u> $(\lambda,\eta(\lambda,\lambda^\times))$ <u>is nuclear</u>. At this stage we also find it convenient to define a normal set in an l.c. TVS $(X,\mathcal{T})$ possessing a Schauder base $\{x_n;f_n\}$. Indeed, a set $B \subset X$ will be called <u>normal</u> provided for each $\alpha = \{\alpha_n\} \in \omega$, with $|\alpha_n| \leq 1$ for $n \geq 1$, we have

$$\sum_{n\geq 1} \alpha_n f_n(x)\, x_n \in B ,$$

whenever $x \in B$. It is clear that the notion of a normal set is really meaningful when the Schauder base is assumed to be bounded multiplier. Thus, if this is the case, for each set $B \subset X$, we define

$$\hat{B} = \{ \sum_{n\geq 1} \alpha_n f_n(x)\, x_n : x \in B, |\alpha_n| \leq 1, \forall\, n \geq 1 \}.$$

Clearly $B \subset \hat{B}$ and that $\hat{B}$ is normal in X. Before we pass on to the first result of this section, we make an observation in the form of

Remark : The strong topology $\beta(X^*,X)$ on X^*, where X is an S-space possessing a bounded multiplier base $\{x_n;f_n\}$, can be considered to be a topology of uniform convergence on some (fundamental) collection of normal bounded sets in X. This follows from the fact that $\hat{B}$ is bounded whenever B is bounded in X. Indeed, from the proof of Proposition 4.2.6(a), we have $\mu = \delta^\times$ so that a set B is bounded in X if and only if it is bounded in $(\delta,\sigma(\delta,\delta^\times))$. And this is true <==> B is bounded in $\eta(\delta,\delta^\times)$ [cf. Theorem 0.2.6 and Proposition 0.3.3] <==> $\hat{B}$ is bounded in $\eta(\delta,\delta^\times)$ (by [65], p. 413) <==> $\hat{B}$ is bounded in $(\delta,\sigma(\delta,\delta^\times))$.

Finally, if X is an l.c. TVS possessing a Schauder base $\{x_n;f_n\}$ and λ an arbitrary sequence space, we denote by $F_a^\pi(\pi \in P(\mathbb{N}), a \in \lambda_+^\times)$, the correspondence

$$F_a^\pi(x) = \sum_{n\geq 1} a_{\pi(n)} f_n(x) x_n, \quad x \in X.$$

F_a^π may not be a well defined mapping. With this background, we are now in a position to state and prove the first result of this section in

Proposition 4.4.1 : Let X be an S-space, having a shrinking and bounded multiplier base $\{x_n;f_n\}$. Consider the following statements:

(1) $\{f_n; Jx_n\}$ is a Q-fully λ-base for $(X^*,\beta(X^*,X))$.

(2) There exists $\pi \in P(\mathbb{N})$ such that $F_a^\pi : X \to X$ is a well defined bounded linear mapping, for each $a \in \lambda_+^\times$.

(3) There exists $\pi \in P(\mathbb{N})$ such that $F_a^\pi : X \to X$ is a well defined continuous linear mapping, $a \in \lambda$

(4) There exists $\pi \in P(\mathbb{N})$ such that $F_a^\pi : X \to X$ is a well defined linear mapping for each $a \in \lambda_+^\times$.

Then the following conclusions are valid :

(a) (3) ==> (2) ==> (1); (2) ==> (4).

(b) For barrelled X, (4) ==> (3) and for a bornological space X, (2) ==> (3).

(c) If X is semireflexive, then (1) ==> (2).

(d) If X is a Fréchet reflexive space, then (1) <==> (2) <==> (3) <==> (4).

Proof: (a), (3) ==> (2). Since continuous linear mappings are bounded, the implication (3) ==> (2) follows trivially.

(2) ==> (1). In view of the remark preceding this result, the topology $\beta(X^*, X)$ on X^* can be thought of as an S-topology where S is the collection of all normal bounded sets in X. By the given hypothesis, we have $B_{\pi,a} \in S$, for $B \in S$ and $a \in \lambda_+^\times$ where

$$B_{\pi,a} = F_a^\pi(B) = \{ \sum_{i \geq 1} f_i(x)\, a_{\pi(i)} x_i : x \in B \}.$$

Let p_B denote the continuous seminorm on X_β^* corresponding to $B \in S$. Hence for $f \in X^*$, we have

$$p_{B_{\pi,a}}(f) = \sup \{|f(x)|: x \in B_{\pi,a}\}$$

$$= \sup_{x\in B} \{|\sum_{i\geq 1} f(x_i)\, a_{\pi(i)} f_i(x)|\}$$

$$= p_B \left[\sum_{i\geq 1} f(x_i)\, a_{\pi(i)} f_i\right].$$

Let $F_n = f(x_n)f_n$, $n\geq 1$. Then we find that

$$p_{B_{\pi,a}}(F_n) = p_B(f(x_n)a_{\pi(n)}f_n), \quad n \geq 1.$$

Next, we observe that if $|f(x_i)| \leq |g(x_i)|$, $i \geq 1$, where $f, g \in X^*$, then for any $B \in \mathcal{S}$, $p_B(f) \leq p_B(g)$. Indeed, for $x \in B$, we have

$$|f(\sum_{i\geq 1} f_i(x)x_i)| \leq \sum_{i\geq 1}|f_i(x)||f(x_i)| \leq \sum_{i\geq 1}|f_i(x)||g(x_i)|$$

$$= |\sum_{i\geq 1} \alpha_i f_i(x) g(x_i)|$$

where $\{\alpha_i\} \in \omega$ with $|\alpha_i| = 1$, $i \geq 1$. Therefore it follows that

$$|f(\sum_{i\geq 1} f_i(x)x_i)| \leq |g(y)|, \; y = \sum_{i\geq 1} \alpha_i f_i(x)x_i \in B .$$

Thus we get

$$|f(x)| \leq |g(y)| \leq p_B(g), \quad \forall\, x \in B$$

$$\Longrightarrow \quad p_B(f) \leq p_B(g).$$

In particular, since $B_{\pi,a} \in \mathcal{S}$ and

$$|F_n(x_i)| \leq |f(x_i)|, \quad \forall\, i \geq 1,$$

we get the inequality

$$p_{B_{\pi,a}}(F_n) \leq p_{B_{\pi,a}}(f)\ , \quad \forall\, n \geq 1.$$

Hence

$$\sup_{n\geq 1} \{p_B[f(x_n)a_{\pi(n)}f_n]\} \leq p_{B_{\pi,a}}(f), \quad \forall\, f \in X^*.$$

$$\Longrightarrow \quad \sup_{n\geq 1} \{|f(x_{\sigma(n)})| p_B(f_{\sigma(n)})a_n\} \leq p_{B_{\pi,a}}(f), \quad \forall\, f \in X^*,$$

where $\sigma = \pi^{-1}$. Invoking Proposition 4.2.7, it follows that $\{f_n; Jx_n\}$ forms a Q-fully λ-base in $(X^*, \beta(X^*,X))$.

(2) ==> (4). This is obvious.

(b). We show that (4) ==> (3), i.e., we show that the mapping $F_a^\pi: X \to X$, which exists by virtue of (4), is continuous for each $a \in \lambda_+^\times$. Consider, for $n \in \mathbb{N}$, the mapping $F_{a,n}^\pi: X \to X$, where $F_{a,n}^\pi(x) = \sum_{i=1}^{n} a_{\pi(i)} f_i(x) x_i$, $x \in X$. Clearly $F_{a,n}^\pi(x) \to F_a^\pi(x)$, for each $x \in X$. Thus, by the Banach-Steinhaus theorem (cf. Theorem 0.2.3), F_a^π is continuous because $F_{a,n}^\pi$ is continuous for each $n \geq 1$.

(2) ==> (3). This follows from the fact that the bounded linear mappings on a bornological space are continuous.

(c). Assume that (1) is true. Thus there exists a permutation $\sigma \in P(\mathbb{N})$ such that $\{f(x_{\sigma(n)})\, p_B(f_{\sigma(n)})\} \in \lambda$,

$$\sum_{n\geq 1} p_B(f_{\sigma(n)})\, a_n|f(x_{\sigma(n)})| \leq p_{B_1}(f),\ \forall\ f \in X^*.$$

$$(*) \implies \sum_{n\geq 1} p_B(f_n)|f(x_n)|a_{\pi(n)} \leq p_{B_1}(f),\quad \forall\ f \in X^*,$$

where $\pi = \sigma^{-1}$. The existence of the mapping F_a^π is guarenteed by an application of a result of Cook (cf. Theorem 0.4.2) which says that under the given hypothesis, $\{x_n; f_n\}$ is boundedly complete. Now $\{\sum_{i=1}^{n} f_i(x)\, a_{\pi(i)}x_i\}$ is a bounded sequence in X for each $x \in X$. Indeed, there exists some $B \in \mathcal{S}$ such that $x \in B$ and, therefore, using (*), the inequality

$$\begin{aligned}|f(\sum_{i=1}^{n} f_i(x)a_{\pi(i)}x_i)| &\leq \sum_{i\geq 1}|f_i(x)a_{\pi(i)}f(x_i)|\\ &\leq \sum_{i\geq 1}p_B(f_i)|a_{\pi(i)}f(x_i)|\\ &\leq p_{B_1}(f),\ f \in X^*\end{aligned}$$

is valid, so that $\{\sum_{i=1}^{n} f_i(x)\, a_{\pi(i)}x_i\}$ is weakly bounded and hence bounded in the original topology of X. Thus it follows that for each $x \in X$, the series

$$\sum_{n\geq 1} a_{\pi(n)}f_n(x)x_n$$

is convergent in X so that $F_a^\pi : X \to X$ is well defined.
To establish the boundedness of F_a^π, let $B \in \mathcal{S}$ and $f \in X^*$ be arbitrarily chosen. Then for $x \in B$, we have

$$|f(F_a^\pi(x)) \leq \sum_{n\geq 1} |f_n(x)|a_{\pi(n)}f(x_n)|$$

$$\leq \sum_{n\geq 1} p_B(f_n)| f(x_n)|a_{\pi(n)} \leq p_{B_1}(f),$$

where the last inequality follows from (*). Therefore,

$$\sup_{x\in B} \{|f(F_a^\pi(x))|\} \leq p_{B_1}(f) ,$$

thereby establishing the boundedness of the mapping $F_a^\pi:X \to X$, for each $a \in \lambda_+^\times$.

(d) Here the desired implications follow from (a) through (c). Thus the result is completely established.

We have seen in the foregoing Proposition that under the same conditions on X as laid down there, the continuity of the maps $F_a^\pi:X \to X$ ($a \in \lambda_+^\times$) where $\pi \in P(\mathbb{N})$, is sufficient to ensure that the dual basis $\{f_n; Jx_n\}$ be a Q-fully λ-base in X_β^*. The question as to when does this condition on F_a^π's yield the same conclusion for the Schauder base $\{x_n;f_n\}$ in X, is answered by the following result for Fréchet spaces. This result will be later used to prove the main theorem of this section.

<u>Proposition</u> 4.4.2 : Let X be a Fréchet space having a bounded multiplier base $\{x_n;f_n\}$. Then $\{x_n;f_n\}$ is a Q-fully λ-base in X if and only if there exists a permutation $\pi \in P(\mathbb{N})$ such that $F_a^\pi:X \to X$ defines a continuous linear mapping, for each $a \in \lambda_+^\times$.

Proof : Assume that the Schauder base $\{x_n;f_n\}$ is a Q-fully λ-base in X. Thus there exists $\sigma \in P(\mathbb{N})$ such that to each $p \in \mathcal{D}$ and $a \in \lambda_+^\times$, there corresponds $q \in \mathcal{E}$ with $\{f_{\sigma(n)}(x)p(x_{\sigma(n)})\} \in \lambda$, for each $x \in X$ and

$$\sum_{n\geq 1} |f_n(x)|p(x_n)a_{\pi(n)} \leq q(x), \quad \forall\, x \in X$$

where $\pi = \sigma^{-1}$. The foregoing inequality yields the existence and continuity of F_a^π because for each $p \in \mathcal{D}$, we have

$$p(F_a^\pi(x)) \leq \sum_{n\geq 1} |f_n(x)|\; p(x_n)a_{\pi(n)} \leq q(x),\ x \in X.$$

For proving the converse, we first outline the existence of a fundamental neighbourhood system at the origin, consisting of normal sets in X. To this end, let $\mathcal{U}_X$ denote as usual, a neighbourhood system at the origin consisting of barrels in X and let $\psi : \ell^\infty \times X \to X$ be a mapping defined by

$$\psi(\alpha,x) = \sum_{n\geq 1} \alpha_n f_n(x)x_n\ , \quad \alpha \in \ell^\infty, \quad x \in X.$$

The existence of ψ is guarenteed by the fact that the base $\{x_n;f_n\}$ in X is bounded multiplier. Using the Banach-Steinhaus theorem and the fact that the mappings

$$F_\alpha^{(n)}: X \to X,\ K_x^{(n)}: \quad \ell^\infty \to X\ ,$$

defined by

$$F_\alpha^{(n)}(x) = \sum_{i=1}^{n} \alpha_i f_i(x)\, x_i, \quad (\alpha \text{ fixed})$$

and

$$K_x^{(n)}(\alpha) = \sum_{i=1}^{n} \alpha_i f_i(x)\, x_i \qquad (x \text{ fixed}),$$

are continuous, one easily establishes the separate continuity of Ψ. Since X is a Fréchet space, it follows by a result in [104], p. 51 that Ψ is (jointly) continuous on $\ell^\infty \times X$. Thus for each $u \in \mathcal{U}_X$, there exists $v \in \mathcal{U}_X$ and $\epsilon > 0$ such that

$$(*) \qquad W \equiv W(\epsilon,v) = \Psi(S_\epsilon, v) \subset u,$$

S_ϵ being the closed unit ball in ℓ^∞ with radius ϵ. Since

$$W(\epsilon,v) = \{\sum_{n\geq 1} \alpha_n f_n(x) x_n : x \in v,\ |\alpha_n| \leq \epsilon,\ \forall\, n \geq 1\}$$

and $v \subset W(\epsilon,v)$, one finds that $W(\epsilon,v)$ is a normal neighbourhood at the origin in X for each $v \in \mathcal{U}_X$ and $\epsilon > 0$. Let

$$\hat{\mathcal{U}} = \{\Gamma W(\epsilon,v) : v \in \mathcal{U}_X,\ \epsilon > 0\},$$

where Γ denotes the balanced convex hull of a set in X. It is clear from (*) that $\hat{\mathcal{U}}$ is a fundamental neighbourhood system at the origin, consisting of absolutely convex sets in X. The fact that the members of $\hat{\mathcal{U}}$ are normal, follows if we show that $\Gamma W(\epsilon,v)$ coincides with its normal hull $[\Gamma W(\epsilon,v)]^\wedge$. Thus, if $z \in [\Gamma W(\epsilon,v)]^\wedge$, then

$$z = \sum_{j\geq 1} \alpha_j f_j(y) x_j,\quad |\alpha_j| \leq 1,\ j \geq 1,\ y \in \Gamma W(\epsilon,v).$$

But

$$y = \sum_{i=1}^{N} \beta_i y_i \text{ with } \sum_{i=1}^{N} |\beta_i| \leq 1 \text{ and } y_i \in W(\varepsilon, v),$$
$$1 \leq i \leq N.$$

Therefore,

$$z = \sum_{j\geq 1} \alpha_j \Big(\sum_{i=1}^{N} \beta_i f_j(y_i) \Big) x_j = \sum_{i=1}^{N} \beta_i \Big(\sum_{j\geq 1} \alpha_j f_j(y_i) x_j \Big)$$
$$= \sum_{i=1}^{N} \beta_i u_i$$

where $u_i = \sum_{j\geq 1} \alpha_j f_j(y_i) x_j \in W(\varepsilon, v)$, by the normality of W.

Hence $z \in \overline{\Gamma W(\varepsilon, v)}$ so that $\overline{\Gamma W(\varepsilon, w)}$ is normal.

Thus we can suppose that the topology of X is generated by the family $\{p_U : U \in \hat{\mathcal{U}}\}$ of (monotone) seminorms p_U on X. Indeed, if for $x \in X$, we let $E(x,U) = \{\lambda > 0 : x \in \lambda U\}$, then for $x, y \in X$, with $|f_i(x)| \leq |f_i(y)|$, $i \geq 1$, it follows that $E(y,U) \subset E(x,U)$ so that $p_U(x) \leq p_U(y)$.

Since $F_a^{\pi} : X \to X$ is continuous for some $\pi \in \mathcal{P}(\mathbb{N})$ and each $a \in \lambda_+^{\times}$, we get, corresponding to each $U \in \hat{\mathcal{U}}$, some $V \in \hat{\mathcal{U}}$ such that

$$p_U(F_a^{\pi}(x)) \leq p_V(x), \quad \forall\, x \in X.$$

Now for each $n \in \mathbb{N}$, we have

$$|f_i(a_{\pi(n)} f_n(x) x_n)| \leq |f_i(F_a^{\pi}(x))|, \quad i \geq 1.$$

Therefore, by the above argument, for each $U \in \hat{\mathcal{U}}$, we have

$$p_U(a_{\pi(n)} f_n(x) x_n) \leq p_U(F_a^{\pi}(x)) \leq p_V(x), \quad \forall\, x \in X \text{ and } n \geq 1,$$

and this gives

$$\sup_{n\geq 1} \{|f_{\sigma(n)}(x)|p_U(x_{\sigma(n)})a_n\} \leq p_V(x), \quad \forall\, x \in X.$$

(Here $\sigma = \pi^{-1}$). Making use of Proposition 4.2.7, we find that $\{x_n; f_n\}$ is a Q-fully λ-base in X and this completes the proof.

We still have one more result before we state and prove the main theorem of this section.

Proposition 4.4.3 : Suppose X is a sequentially complete l.c. TVS equipped with a fully-λ-base $\{x_n; f_n\}$, where λ satisfies the (K)-property. Then X is a nuclear space. (Here λ need not be perfect).

Proof: By Proposition 4.3.1, X can be topologically identified with $\Lambda(P_o)$, where $P_o = \{\{p(x_n)a_n\}: p \in \mathcal{D}, a \in \lambda^\times_+\}$. Thus X is nuclear if and only if $\Lambda(P_o)$ is nuclear (relative to its Köthe topology $\widetilde{\mathcal{T}}_{P_o}$). Since $(\lambda, \eta(\lambda, \lambda^\times))$ is assumed to be nuclear, for each $a \in \lambda^\times_+$ one can find $b \in \lambda^\times_+$, $b \geq a$ such that $\sum_{n\geq 1} a_n/b_n < \infty$ (cf. Proposition 0.5.7). Consider an arbitrary $p \in \mathcal{D}$ and $a \in \lambda^\times_+$. Then taking b=a and noting that the sequences $\{p(x_n)a_n\}$ and $\{p(x_n)b_n\}$ belong to P_o, it follows from the above mentioned result that $\Lambda(P_o)$ is nuclear.

Remark : Note that the preceding proposition is true for any sequence space $\lambda \subset \ell^1$ whenever λ is nuclear in its normal

topology $\eta(\lambda,\lambda^{\times})$. However, it turns out that the nuclearity of $(\lambda,\eta(\lambda,\lambda^{\times}))$ is not necessarily implied by the nuclearity of a sequentially complete l.c. TVS X possessing a fully-λ-base. In fact any nuclear Fréchet space with a basis would testify. It would, therefore, be interesting to discover the class of sequence spaces $\lambda \subsetneq \ell^1$ for which the converse of the foregoing result fails to be true.

We are now ready to pass on to the main result of this chapter, namely

<u>Theorem</u> 4.4.4 : Let λ be a sequence space satisfying the (K)-property. Then a Schauder base $\{x_n;f_n\}$ in a Fréchet space X is a Q-fully λ-base if and only if $\{f_n; Jx_n\}$ is a Q-fully λ-base in $(X^*, \beta(X^*,X))$.

<u>Proof</u> : Assume first that $\{x_n;f_n\}$ is a Q-fully λ-base for X. Thus for some $\sigma \in P(\mathbb{N})$, $\{x_{\sigma(n)}, f_{\sigma(n)}\}$ is a fully-λ-base for X. By the foregoing result, X is a nuclear space and hence reflexive in virtue of Proposition 0.5.6. Applying Cook's result, referred to earlier, we conclude that $\{x_n;f_n\}$ is shrinking and boundedly complete. Also, since $\lambda \subset \ell^1$ and $\{x_n;f_n\}$ is a semi-λ-base, it follows that $\{x_n;f_n\}$ is absolute and hence a bounded multiplier base in X. An application of Proposition 4.4.2 yields the existence and continuity of the map F_a^{π} for each $a \in \lambda_+^{\times}$ and some $\pi \in P(\mathbb{N})$. Thus by Proposition 4.4.1, (2) ==> (1), we find that $\{f_n;Jx_n\}$ is a Q-fully λ-base for $(X^*,\beta(X^*,X))$.

Conversely, there exists some $\sigma \in P(\mathbb{N})$ such that $\{f_{\sigma(n)}, Jx_{\sigma(n)}\}$ is a fully-λ-base for $(X^*,\beta(X^*,X))$. Since $(X^*,\beta(X^*,X))$ is (sequentially) complete, it is nuclear by Proposition 4.4.3, so that X itself is nuclear by Proposition 0.5.5. Proceeding along the same lines of argument as in the necessity part, we find that X is reflexive and that the base $\{x_n;f_n\}$ is shrinking and bounded multiplier. Applying Proposition 4.4.1, (1) ==> (3), there follows the continuity of the map $F_a^{\pi}:X \to X$, $a \in \lambda_+^{\times}$ where π is some permutation in $P(\mathbb{N})$. Finally, invoking Proposition 4.4.2, we conclude that $\{x_n;f_n\}$ is a Q-fully λ-base in X. This completes the proof of the theorem.

<u>Remark</u> : An inspection of the preceding results of this section allows us to state the following variant of the foregoing theorem.

<u>Theorem</u> 4.4.5 : Assume that λ satisfies the conditions of Theorem 4.4.4. Then a Schauder base $\{x_n;f_n\}$ in a Fréchet space X is a fully-λ-base if and only if $\{f_n;Jx_n\}$ is a fully-λ-base in $(X^*,\beta(X^*,X))$.

<u>Note</u> : De Grande-De Kimpe [24] has obtained the same result for reflexive l.c. TVS X.

<u>Remark</u> : Imitating the proofs of Propositions 4.4.1 through 4.4.3 and those of Theorems 4.4.4 and 4.4.5, we can prove the following result where $(\lambda,\eta(\lambda,\lambda^{\times}))$ is replaced by $(\Lambda(P),\mathcal{T}_P)$.

<u>Theorem</u> 4.4.6 : Let $\Lambda(P)$ be a Köthe space such that $\Lambda(P) \subset \ell^1$ and that $(\Lambda(P), \mathcal{T}_P)$ is nuclear. Further, let X be a Fréchet space possessing a Schauder base $\{x_n; f_n\}$. Then $\{x_n; f_n\}$ is a fully-$\Lambda(P)$-base (resp. Q-fully $\Lambda(P)$-base) in X if and only if $\{f_n; Jx_n\}$ is a fully-$\Lambda(P)$-base (resp. Q-fully $\Lambda(P)$-base) in $(X^*, \beta(X^*, X))$.

<u>Proof</u> : The proof follows by modifying Proposition 4.3.1 to the case when $(\lambda, \eta(\lambda, \lambda^\times))$ is replaced by $(\Lambda(P), \mathcal{T}_P)$.

As a special case of the foregoing theorem, we obtain the following result proved earlier in [31].

<u>Corollary</u> 4.4.7 : Let α be a nuclear exponent sequence of infinite type. Then a base $\{x_n; f_n\}$ in a Fréchet space X is a Q-fully $\Lambda_\infty(\alpha)$-base if and only if the dual base $\{f_n; Jx_n\}$ is a Q-fully $\Lambda_\infty(\alpha)$-base in $(X^*, \beta(X^*, X))$.

The last theorem of this section makes use of the following results proved in [32], p. 40 and [99], Theorem 4.4, respectively.

(A): Let λ be a sequence space such that $\beta(\lambda, \lambda^\times)$ is compatible with $\langle \lambda, \lambda^\times \rangle$. Then a Köthe space $\Lambda(P)$ is nuclear if it is λ-nuclear.

(B): Let $\Lambda = \bigcap\limits_{k \geq 1} \frac{1}{a^k} \ell^1$ be a nuclear Köthe space such that $\Lambda(P)$ is Λ-nuclear. Then $\Lambda(P)$ is strongly nuclear and a <u>fortiori</u> nuclear.

We end up this chapter with

<u>Theorem</u> 4.4.8 : Let $\Lambda(P) \subset \ell^1$ be such that (a) $\Lambda(P)$ is λ-nuclear where $(\lambda, \beta(\lambda, \lambda^\times))^* = \lambda^\times$ or (b) $\Lambda(P)$ is λ-nuclear where $\Lambda = \bigcap_{k \geq 1} \frac{1}{a^k} \ell^1$ is a nuclear Köthe space. Then a Schauder base $\{x_n; f_n\}$ in a Fréchet space X is a $\Lambda(P)$-base if and only if $\{f_n; Jx_n\}$ is a $\Lambda(P)$-base in $(X^*, \beta(X^*, X))$.

<u>Proof</u> : The simple proof which makes use of Proposition 4.4.3, is omitted.

CHAPTER 5

λ-BASES AND λ-NUCLEARITY

1. Introduction : It is well known that the theory of Schauder bases is particularly well adapted when applied to nuclear spaces - the veracity of this statement is amply testified by the outstanding work of several authors, notably by those of Dynin and Mityagin [33], Mityagin [73] and Wojtyński [124] (cf. also [50]), to name only a few. (For more historical remarks, see Chapter 1). The question as to whether the richness of Schauder bases in nuclear spaces can be preserved in the more general setting of λ-nuclear spaces, forms the subject matter of this chapter. In fact, it is shown that though there exists a partial analogue of the Dynin-Mityagin theorem in case of λ-nuclear spaces, a natural analogue of the Pietsch's theorem (on absolute bases in nuclear spaces) fails to hold in the context of λ-bases and λ-nuclear spaces (see Example 5.2.4). This situation is demonstrated in Section 2 of this chapter, while Section 3 deals with the characterisation of $\hat{\Lambda}(P;\phi)$-nuclear spaces having a Schauder base. The chapter concludes with some consequences of this result.

2. $\Lambda(P)$-Bases And $\Lambda(P)$-Nuclear Spaces : In this section we study the relationship between the λ-bases and λ-nuclearity

of a space and then attempt to answer the questions motivated by those known from the theory of Schauder bases in nuclear spaces. This study entails an investigation into the impact of λ-nuclearity on the bases in a locally convex space and vice versa. To be more precise in formulating the problem and seeking its solution, we recall the following two results from the theory of (bases in) nuclear spaces, which reflect the impact of an absolute Schauder base (in a Fréchet space) that it carries on the nuclear structure of the locally convex space in question.

Theorem A (Dynin and Mityagin [33]; Wojtyński [124]): A Fréchet space X with a Schauder base is nuclear if and only if each base in X is absolute.

Theorem B (Pietsch [88]) : A Fréchet space with an absolute base $\{x_n;f_n\}$ is nuclear if and only if $\{f_n;Jx_n\}$ is an absolute base in $(X^*,\beta(X^*,X))$, J being the natural (canonical) embedding from X into X^{**}.

In view of the terminology explained in Chapter 4, we note that in Theorems A and B above, the terms ''absolute'' and ''nuclear'' can be replaced by ''fully-ℓ^1-absolute'' and ''ℓ^1-nuclear'', respectively. Now the problem we are interested to formulate is simple. Indeed, we would like to know the class of sequence spaces λ which could replace the space ℓ^1 in the aforementioned two theorems. Insofar

as Theorem A is concerned, there exists only a partial analogue of this result in the present more general setting. The proof of this statement is contained in

<u>Theorem</u> 5.2.1 : Let $\Lambda(P)$ be a stable nuclear G_∞-space. Then every Schauder base $\{x_n; f_n\}$ in a $\Lambda(P)$-nuclear Fréchet space X is a Q-fully $\Lambda(P)$-base.

<u>Proof</u> : Since X is $\Lambda(P)$-nuclear, it is nuclear because $\Lambda(P) \subset \ell^1$. By Theorem A, $\{x_n; f_n\}$ is a fully-absolute base in X. Hence by [89], p. 172, X can be topologically identified with a Köthe space $\Lambda(P_o)$ where $P_o = \{(p_k(x_n))_n;\ k \geq 1\}$ – $\{p_k;\ k \geq 1\}$ being a generating sequence of seminorms for the topology of X. Thus it suffices to show that $\{e^n; e^n\}$, which is a Schauder base for $\Lambda(P_o)$ [see the remarks preceding Proposition 0.4.4], is a Q-fully $\Lambda(P)$-base for $\Lambda(P_o)$. By the given hypothesis, $\Lambda(P_o)$ is $\Lambda(P)$-nuclear and, therefore, by a result in [96], Proposition 4.9, there exists $\pi \in \mathcal{P}(\mathbb{N})$ such that for each $k \geq 1$, we can find $m \geq k$ satisfying

$$(*) \qquad \left\{\frac{p_k(x_{\pi(n)})}{p_m(x_{\pi(n)})}\right\} \in \Lambda(P).$$

To prove the required assertion, let $k \geq 1$ and $a \in P$ be chosen arbitrarily. Then for $y \in \Lambda(P_o)$, we have

$$|\langle y, e^{\pi(n)}\rangle| p_k(x_{\pi(n)})\, a_n = |y_{\pi(n)}| a_n p_m(x_{\pi(n)}) \left[\frac{p_k(x_{\pi(n)})}{p_m(x_{\pi(n)})}\right].$$

Hence

$$\sum_{n\geq 1} |\langle y,e^{\pi(n)}\rangle| p_k(x_{\pi(n)})\, a_n \leq \Big(\sum_{n\geq 1} |y_n| p_m(x_n)\Big)\Big(\sum_{n\geq 1} \frac{p_k(x_{\pi(n)})}{p_m(x_{\pi(n)})} a_n\Big)$$

$$\leq K_a \sum_{n\geq 1} |y_n| p_m(x_n) = K_a \hat{p}_m(y),$$

where

$$K_a = \sum_{n\geq 1} \frac{p_k(x_{\pi(n)})}{p_m(x_{\pi(n)})} a_n < \infty, \text{ by } (*).$$

Since the above inequality is satisfied for each $y \in \Lambda(P_o)$, we are led to the assertion that $\{\langle y,e^{\pi(n)}\rangle\, \hat{p}_k(e^{\pi(n)})\} \in \Lambda(P_o)$ for each $y \in \Lambda(P_o)$ so that $\{e^n;e^n\}$ is a Q-fully $\Lambda(P)$-base for $\Lambda(P_o)$.

As a partial converse to the above theorem, we have

<u>Theorem</u> 5.2.2 : Let X be a sequentially complete l.c. TVS having a fully-$\Lambda(P)$-base, P being a G_∞-Köthe set. Then X is $\hat{\Lambda}(P_o;\phi)$-nuclear for each nuclear Köthe space $\Lambda(P_o)$ of increasing type such that $\Lambda(P)$ is $\hat{\Lambda}(P_o;\phi)$-nuclear.

<u>Proof</u> : The result follows by applying Theorem 3.2.4 and Proposition 4.3.5 and its proof runs on lines parallel to those of Proposition 4.4.3.

Combining Theorems 5.2.1 and 5.2.2, we obtain

<u>Theorem</u> 5.2.3 : Let X be a Fréchet space having a fully-$\Lambda(P)$-base, $\Lambda(P)$ being a G_∞-space which is $\Lambda(P_o)$-nuclear for some stable nuclear G_∞-set P_o. Then every Schauder base in X is a Q-fully $\Lambda(P_o)$-base.

As regards Theorem B, it turns out that the same breaks down when ℓ^1 is replaced by a nuclear G_∞-space $\Lambda(P)$. This shows that there is a strong deviation from the classical theory of bases in nuclear spaces when one considers the corresponding situation of λ-bases in λ-nuclear spaces. This aspect of the situation is demonstrated in the following counter-example.

<u>Example</u> 5.2.4 : Let $X = \Lambda(P)$, where P is a countable nuclear G_∞-set so that X is a nuclear Fréchet space. It is clear that $\{e^n; e^n\}$ forms a Schauder base for X. The fact that $\{e^n; e^n\}$ is a fully-$\Lambda(P)$-base for X, follows in view of the G_∞-character of P, which yields, corresponding to each $a, b \in P$, some element $c \in P$ satisfying $a_n b_n \leq c_n$, $n \geq 1$. Therefore, for each $x \in X$, we have

$$|\langle x, e^n\rangle| p_a(e^n) b_n = |x_n| a_n b_n \leq |x_n| c_n, \; n \geq 1$$

which leads to the inequality

$$\sum_{n \geq 1} |\langle x, e^n\rangle| p_a(e^n) b_n \leq \sum_{n \geq 1} |x_n| c_n < \infty ,$$

so that $\{\langle x, e^n\rangle p_a(e^n)\} \in \Lambda(P)$ and, therefore, $\{e^n; e^n\}$ is a fully-$\Lambda(P)$-base for X. We now show that $\{e^n; e^n\}$ forms a fully-$\Lambda(P)$-base for $(X^*, \beta(X^*, X))$. To this end, let B be any bounded set in $X \equiv \Lambda(P)$, then by a result in [67], there exists $y \in X$ such that

$$\sup_{x \in B} |x_n| \leq |y_n|, \quad n \geq 1.$$

Now for $f \in X^*$, we have (cf. [37], p. 965 and [58], p. 60) $f \equiv \{f(e^n)\} = \{f_n\}$, say and, therefore, the inequality

$$p_B(e^n)|\langle f, e^n\rangle| = \sup_{x \in B} |x_n||f_n| \leq \rho|y_n||c_n|,$$

is satisfied for each $n \geq 1$. Here $c \in P$ and $\rho > 0$ such that $|f_n| \leq \rho|c_n|$, $n \geq 1$. Thus if $a \in P$, we get

$$\sum_{n \geq 1} p_B(e^n)|\langle f, e^n\rangle| a_n \leq \rho \sum_{n \geq 1} |y_n||a_n c_n|$$
$$\leq \rho \sum_{n \geq 1} |y_n b_n| < \infty,$$

where $b \in P$ is such that $a_n c_n \leq b_n$, $n \geq 1$. Since X is Fréchet nuclear, it is reflexive by Proposition 0.5.6 and, therefore, $(X^*, \beta(X^*, X))$ is reflexive and a fortiori barrelled. Now the above inequality suggests that $\{e^n; e^n\}$ is a semi-$\Lambda(P)$-basis in $(X^*, \beta(X^*, X))$ so that it is also a fully-$\Lambda(P)$-basis, by virtue of Proposition 4.2.6(a). (The fact that $\{e^n; e^n\}$ is a fully-$\Lambda(P)$-base for X^*_β also follows as a direct consequence of Theorem 4.4.6). However, by Proposition 3.2.4, the space X cannot be $\Lambda(P)$-nuclear!

3. Schauder Bases In $\hat{\Lambda}(P;\phi)$-Nuclear Spaces : It is well known (see [40], [88] and [50]; cf. also [52]) that an l.c. TVS X with an equicontinuous base $\{x_n; f_n\}$ is nuclear if and only if for each $p \in \mathcal{D}$, there exists $q \in \mathcal{D}$, $q \geq p$ such that

$$\sum_{n\geq 1} \frac{p(x_n)}{q(x_n)} < \infty \quad (\tfrac{0}{0} = 0).$$

The above result is easily seen to be equivalent to the 'absolute basis theorem' of Dynin and Mityagin [33] which states that each equicontinuous base in a nuclear space is absolute. As against the partial truth of the theorem of Dynin and Mityagin and the falsity of the theorem of Pietsch in $\hat{\lambda}$-nuclear spaces (see the preceding section), we show below that the above cited equivalence can be retained in case of $\hat{\lambda}$-nuclearity associated with the sequence space $\lambda = \Lambda(P_0; \phi)$, where P_0 is a nuclear Köthe set of infinite type. In what follows, the symbol P_0 will denote a Köthe set of this type, unless stated otherwise. Before we come to the main result of this section, let us make the following observation in the form of

Remark 5.3.1 : If P_0 is a nuclear G_∞-set and ϕ is a function as defined in Section 3 of Chapter 0, then it follows that $\{nx_n\} \in \Lambda(P_0; \phi)$ whenever $x \in \Lambda(P_0; \phi)$. Indeed, using the nuclearity of $\Lambda(P_0)$, there exists $k \geq 1$ and $b \in P_0$ such that $n \leq kb_n$, $n \geq 1$. Also for $a \in P_0$, there exists $c \in P_0$ such that $a_n b_n \leq c_n$, $n \geq 1$. Thus, if $a \in P_0$, we get

$$\sum_{n\geq 1} \phi(|nx_n|)a_n \leq \sum_{n\geq 1} n\phi(|x_n|)\, a_n \leq k \sum_{n\geq 1} \phi(|x_n|)c_n < \infty$$

$$\Longrightarrow \{nx_n\} \in \Lambda(P_0; \phi).$$

The main result of this section is contained in

Theorem 5.3.2 : An l.c. TVS X equipped with an equicontinuous base $\{x_n; f_n\}$ is $\hat{\Lambda}(P_o;\phi)$-nuclear if and only if to each $p \in \mathcal{D}$, there correspond $q \in \mathcal{D}$ and an injection $\pi: \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N}: p(x_n) \neq 0\}$ such that

$$(*) \qquad \left\{\frac{p(x_{\pi(n)})}{q(x_{\pi(n)})}\right\} \in \Lambda(P_o;\phi).$$

Proof (Necessity) : Since X is $\hat{\Lambda}(P_o;\phi)$-nuclear, it is nuclear and a fortiori a Schwartz space. Hence from [115], p. 8, we get for each $p \in \mathcal{D}$, some $q_1 \in \mathcal{D}$, $q_1 \geq p$ such that

$$\text{(i)} \qquad \frac{p(x_n)}{q_1(x_n)} \to 0$$

Set

$$p^*(x) = \sup_{n\geq 1} \{|f_n(x)|p(x_n)\}, \; x \in X.$$

Then the equicontinuity of $\{x_n; f_n\}$ yields the continuity of p^* on X. Hence, using Theorem 2.3.4, we get $q_2 \in \mathcal{D}$ such that

$$\{\delta_n(u_{q_2}, u_{p^*})\} \in \Lambda(P_o;\phi)$$

$$\text{(ii)} \qquad \Longrightarrow \{n\delta_n(u_{q_2}, u_{p^*})\} \in \Lambda(P_o;\phi),$$

by the remark preceding this theorem. Here $u_{q_1} = \{x \in X; q_1(x) \leq 1\}$.

Also, there exists $q_3 \in \mathcal{D}$ such that

(iii) $$p^*(x) \leq q_3(x), \quad \forall\, x \in X.$$

Thus if $q(x) = \max\{q_1(x), q_2(x), q_3(x)\}$, then (i), (ii) and (iii) lead to

(1) $$\frac{p(x_n)}{q(x_n)} \to 0$$

(2) $$\{n\delta_n(u_q, u_{p*})\} \in \Lambda(P_o; \phi)$$

(3) $$p^*(x) \leq q(x), \quad \forall\, x \in X.$$

Define

$$\alpha_n = \begin{cases} \dfrac{p(x_n)}{q(x_n)}, & q(x_n) \neq 0 \;; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\alpha_n \to 0$, there exists an injection $\pi: \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N} : p(x_n) \neq 0\}$ and

$$\alpha_{\pi(m)} \leq \alpha_{\pi(n)}, \text{ whenever } m \geq n.$$

Suppose

$$L_m = \operatorname{sp}\{x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(m)}\}.$$

Clearly $\dim L_m \leq m$. Let

$$x \in \alpha_{\pi(m)}(u_{p*} \cap L_m)$$

$$\Longrightarrow \quad x = \sum_{i=1}^{m} f_{\pi(i)}(x)\, x_{\pi(i)}$$

and, therefore,

$$q(x) \le \sum_{i=1}^{m} |f_{\pi(i)}(x)|\, q(x_{\pi(i)})$$

$$= \sum_{i=1}^{m} |f_{\pi(i)}(x)|\, p(x_{\pi(i)}) \cdot \frac{q(x_{\pi(i)})}{p(x_{\pi(i)})}$$

$$\le \alpha_{\pi(m)}^{-1} \sum_{i=1}^{m} |f_{\pi(i)}(x)|\, p(x_{\pi(i)})$$

$$\le \alpha_{\pi(m)}^{-1}\, m \sup_{n\ge 1} \{ |f_n(x)|\, p(x_n) \}$$

$$= m\, \alpha_{\pi(m)}^{-1}\, p^*(x) \le m$$

$$\Longrightarrow \quad x \in m\, u_q$$

$$\Longrightarrow \quad \frac{\alpha_{\pi(m)}}{m} (u_{p^*} \cap L_m) \subset u_q .$$

Therefore, applying Theorem 0.6.12, we get

$$m\delta_m(u_q, u_{p^*}) \ge \alpha_{\pi(m)} \;, \; m \ge 1.$$

Thus from (2), we obtain $\{\alpha_{\pi(m)}\} \in \Lambda(P_o;\phi)$

$$\Longrightarrow \quad \left\{\frac{p(x_{\pi(n)})}{q(x_{\pi(n)})}\right\} \in \Lambda(P_o;\phi).$$

(<u>Sufficiency</u>) : Suppose that (*) holds for a given $p \in \mathcal{D}$, some $q \in \mathcal{D}$ and an injection $\pi : \mathbb{N} \to \mathbb{N}$.

Define

$$g_n(x) = p(x_n)\, f_n(x), \; x \in X$$

and

$$\zeta_n = \sup \{ |g_n(x)| : q^*(x) \le 1 \} \;, \; n \ge 1.$$

Now, for $x \in X$ with $q^*(x) \le 1$, we have

$$\sup_{n\geq 1} \{|f_n(x)|\, q(x_n)\} \leq 1$$

$$\Longrightarrow \quad \sup\{|f_n(x)| : q^*(x) \leq 1\} \leq \frac{1}{q(x_n)}, \quad \forall\, n \geq 1$$

$$\Longrightarrow \quad p(x_n) \sup\{|f_n(x)| : q^*(x) \leq 1\} \leq \frac{p(x_n)}{q(x_n)}, \quad \forall\, n \geq 1$$

$$\Longrightarrow \quad \zeta_{\pi(n)} \leq \frac{p(x_{\pi(n)})}{q(x_{\pi(n)})}, \quad \forall\, n \geq 1 .$$

Therefore, by the hypothesis, we get $\{\zeta_{\pi(n)}\} \in \Lambda(P_o; \phi)$.

If for $x \in X$, we set

$$\hat{g}_n(x) = \begin{cases} \dfrac{g_n(x)}{\zeta_n}, & \text{if } \zeta_n \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

then from

$$x = \sum_{n\geq 1} f_n(x) x_n, \quad x \in X$$

we get

$$\begin{aligned} p(x) &\leq \sum_{n\geq 1} |f_n(x)|\, p(x_n) \\ &= \sum_{n\geq 1} |g_n(x)| = \sum_{n\geq 1} |\zeta_n \langle x, \hat{g}_n\rangle| \\ &= \sum_{n\geq 1} |\zeta_{\pi(n)} \langle x, \hat{g}_{\pi(n)}\rangle|, \quad x \in X. \end{aligned}$$

Since $|g_n(x)|/\zeta_n \leq q^*(x)$, for each $x \in X$ and $n \geq 1$, it follows that the sequence $\{\hat{g}_n\} \subset X^*$ is equicontinuous, and, therefore, invoking Theorem 2.3.4, the $\hat{\Lambda}(P_o; \phi)$-nuclearity of X is established. This completes the proof.

We now apply the foregoing result to obtain new information on the $\hat{\Lambda}(P_0; \phi)$-nuclearity of a locally convex space. The first such application furnishes an alternative proof of the Grothendieck-Pietsch-Köthe criterion for the $\hat{\Lambda}(P_0; \phi)$-nuclearity of a Köthe sequence space, proved earlier in Chapter 3, where P_0 is taken to be a nuclear Köthe set of increasing type (and not a G_∞-set).

<u>Theorem</u> 5.3.3 : A Köthe sequence space $\Lambda(P)$ is $\hat{\Lambda}(P_0;\phi)$-nuclear if and only if to each $a \in P$, there correspond $b \in P$, $b \geq a$ and an injection $\pi\colon \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N} : a_n \neq 0\}$ such that

$$\left\{\frac{a_{\pi(n)}}{b_{\pi(n)}}\right\} \in \Lambda(P_0;\phi).$$

<u>Proof</u> : The natural topology $\mathcal{T}_P$ on $\Lambda(P)$ is generated by the family $\{p_a : a \in P\}$ of seminorms p_a on $\Lambda(P)$, where p_a is defined by

$$p_a(x) = \sum_{n\geq 1} |x_n| a_n, \; x \in \Lambda(P).$$

It is easy to see that $\{e^n; e^n\}$ forms an equicontinuous Schauder base in $\Lambda(P)$. Indeed, for each $x \in \Lambda(P)$ and $a \in P$, we have the inequality

$$\sup_{n\geq 1} \{ |e^n(x)| p_a(e^n) \} = \sup_{n\geq 1} |x_n a_n| \leq p_a(x)$$

which yields the equicontinuity of the base $\{e^n; e^n\}$. Also since

$$\frac{a_{\pi(n)}}{b_{\pi(n)}} = \frac{p_a(e^{\pi(n)})}{p_b(e^{\pi(n)})} \quad , \quad \forall\, n \geq 1 ,$$

the result follows immediately from Theorem 5.3.2.

The proof of the following result runs on the lines similar to those of the preceding theorem.

Theorem 5.3.4 : A sequence space $(\lambda, \mathcal{T}_S)$ equipped with a normal topology $\mathcal{T}_S$ is $\hat{\Lambda}(P_o; \phi)$-nuclear if and only if for each $s \in S$, there exist $s' \in S$ and an injection $\pi : \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N} : \exists\, x \in s$ with $x_n \neq 0\}$ such that

$$\left\{\frac{\sup_{x \in s} |x_{\pi(n)}|}{\sup_{y \in s'} |y_{\pi(n)}|}\right\} \in \Lambda(P_o; \phi).$$

If S is taken to consist of normal covers of singletons $a \in \lambda^{\times}_{+}$ in $\lambda^{\times}$, we get from the above result the following theorem which generalizes an earlier result of Kamthan and Gupta [58], p. 288 where a similar result is proved for the nuclearity of $(\lambda, \eta(\lambda, \lambda^{\times}))$.

Theorem 5.3.5 : A sequence space $(\lambda, \eta(\lambda, \lambda^{\times}))$ is $\hat{\Lambda}(P_o; \phi)$-nuclear if and only if to each $a \in \lambda^{\times}_{+}$, there correspond $b \in \lambda^{\times}_{+}$ and an injection $\pi : \mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N} : a_n \neq 0\}$ such that

$$\left\{\frac{a_{\pi(n)}}{b_{\pi(n)}}\right\} \in \Lambda(P_o; \phi).$$

The following proposition provides an alternative proof of the 'Dynin-Mityagin' theorem for bases in $\Lambda(P)$-nuclear spaces proved earlier in this chapter via the assumption that X is a Fréchet space and that $\Lambda(P)$ is stable. It turns out that these conditions on X and $\Lambda(P)$ can be dispensed with. More precisely, we have

Proposition 5.3.6 : Every equicontinuous Schauder base in a $\Lambda(P_o)$-nuclear space X is a Q-fully $\Lambda(P_o)$-base.

Proof : The given condition together with Theorem 5.3.2 suggest that for each $p \in \mathcal{D}$, there exist $q \in \mathcal{D}$ and an injection $\pi:\mathbb{N} \to \mathbb{N}$ with $\pi(\mathbb{N}) = \{n \in \mathbb{N}: p(x_n) \neq 0\}$ such that

$$\left\{\frac{p(x_{\pi(n)})}{q(x_{\pi(n)})}\right\} \in \Lambda(P_o)$$

$$\Longrightarrow \qquad p(x_{\pi(n)}) = \alpha_n\, q(x_{\pi(n)}), \quad \forall\, n \geq 1,$$

where $\alpha \in \Lambda(P_o)$. Now, for $x \in X$ and $a \in P_o$, we have

$$\sum_{n\geq 1} p(x_{\pi(n)})|f_{\pi(n)}(x)|a_n = \sum_{n\geq 1} |f_{\pi(n)}(x)|\,q(x_{\pi(n)})\alpha_n a_n$$

$$\leq \sup_{n\geq 1}\{|f_n(x)|\,q(x_n)\} \sum_{n\geq 1} |\alpha_n a_n|$$

$$\Longrightarrow \quad \sum_{n\geq 1} p(x_{\pi(n)})|f_{\pi(n)}(x)|a_n \leq M_a q^*(x), \quad \forall\, x \in X,$$

where $M_a = \sum_{n\geq 1}|\alpha_n a_n| < \infty$. The above inequality suggests that $\{x_n;f_n\}$ is a Q-fully $\Lambda(P_o)$-base in X and this completes the proof.

The final result of this chapter which is an improvement upon Theorem 5.2.2 of the preceding section, is contained in

Proposition 5.3.7 : Let P be a Köthe set of increasing type. An l.c. TVS X having a Q-fully $\Lambda(P)$-base $\{x_n; f_n\}$, is $\hat{\Lambda}(P_0;\phi)$-nuclear whenever $\Lambda(P)$ is $\hat{\Lambda}(P_0;\phi)$-nuclear.

Proof : Since $\Lambda(P)$ is assumed to be $\hat{\Lambda}(P_0;\phi)$-nuclear, invoking the proof of Theorem 3.2.4, we get an $a \in P$ with $\{1/a_n\} \in \Lambda(P_0;\phi)$. Now, for any $p \in \mathcal{D}_X$, there exist $q \in \mathcal{D}_X$ and some permutation $\pi \in \mathcal{P}(\mathbb{N})$ such that

$$(*) \qquad \sum_{n\geq 1} p(x_{\pi(n)})\,|f_{\pi(n)}(x)|\,a_n \leq q(x), \quad \forall \; x \in X.$$

In particular, we have

$$p(x_{\pi(n)})\, a_n \leq q(x_{\pi(n)}), \quad \forall\; n \geq 1,$$

$$\Longrightarrow \quad \frac{p(x_{\pi(n)})}{q(x_{\pi(n)})} \leq \frac{1}{a_n}, \quad \forall \; n \geq 1.$$

Therefore

$$\left\{\frac{p(x_{\pi(n)})}{q(x_{\pi(n)})}\right\} \in \Lambda(P_0;\phi).$$

By $(*)$ and the increasing character of P, it follows that $\{x_n;f_n\}$ is an equicontinuous base in X. The last two conditions together with Theorem 5.3.2 imply that X is $\hat{\Lambda}(P_0;\phi)$-nuclear and this completes the proof.

In view of the observation preceding Theorem 2.3.9, it follows that if P is taken to be a nuclear G_∞-set and that ϕ satisfies the condition (Δ), then all the results involving $\hat{\Lambda}(P;\phi)$-nuclearity, established in the previous chapters remain valid if $\hat{\Lambda}(P;\phi)$-nuclearity is replaced by $\Lambda(P;\phi)$-nuclearity. This follows from a result of Nelimarkka [79], p. 30, which says that under these conditions on P and ϕ, the two notions of λ-nuclearity and $\hat{\lambda}$-nuclearity coincide.

REFERENCES

[1] Alpseymen, M. : A generalization of Dragilev's theorem; Jour. reine angew. Math. 276 (1975), 124-129.

[2] ____________ : Basic sequences in some nuclear Köthe sequence spaces: Thesis, University of Michigan (1978).

[3] Apiola, H. : Some permanence properties of locally convex spaces defined by some norm ideals of operators; Stud. Math. 55(1976), 265-278.

[4] ____________ : On multiplicatively stable smooth sequence spaces and permanence of the corresponding λ-nuclearity; Report HTKK-MAT-A84 (1976), 1-14.

[5] ____________ : On tensor products of locally convex A-spaces with applications to λ-nuclearity; Report HTKK-MAT-A115 (1977), 1-32.

[6] ____________ : Characterisation of subspaces and quotients of nuclear $L_f(\alpha,\infty)$-spaces; (preprint).

[7] Bennet, G. and Cooper, J.B. : LF-spaces with absolute bases; Proc. Camb. Philos. Soc. 67(1970), 283-286.

[8] Bessaga, C. : Some remarks on Dragilev's theorem; Stud. Math. 31(1968), 307-318.

[9] ____________ : Nuclear Fréchet spaces without bases I; Bull. Acad. Polon. Sci. 24 (1976), 471-473.

[10] Bessaga, C. and Dubinsky, E. : Nuclear Fréchet spaces without bases III; Arch. der Math. 31(1978), 597-604.

[11] Bessaga, C., Pelczyński, A., and Rolewicz, S. : On diametral approximative dimension and linear homogeneity of F-spaces; Bull. Acad. Polon. Sci. 9(1961), 677-683.

[12] Bessaga, C. and Retherford, J.R. : Lectures on nuclear spaces; Lecture Notes, University of Louisiana.

[13] Boland, P.J. : Holomorphic functions on nuclear spaces; Trans. Amer. Math. Soc. 209 (1975), 278-281.

[14] ________ : An example of a nuclear space in infinite dimensional holomorphy; Arkiv for Matematik 15(1977), 87-91.

[15] Brudovski, V.S. : Associated nuclear topology, mappings of type s and strongly nuclear spaces; Sov. Math. Dokl. 9(1968), 61-63.

[16] ________ : s-type mappings of locally convex spaces; Sov. Math. Dokl. 9(1968), 572-574.

[17] Cắc, Phoung N. : Sur les espaces parfait de suites generalises; Math. Ann. 171 (1967), 131-143.

[18] Ceítlin, Ya. M. : On spaces with absolute bases; Siber. Math. J. 12(5), (1971), 838-841.

[19] Cook, T.A. : Schauder decompositions and semireflexive spaces; Math. Ann. 182(1969), 232-235.

[20] Day, M.M. : Normed Linear Spaces; Springer Verlag, Berlin-Heidelberg-New York (1973).

[21] De Grande-De Kimpe, N. : Λ-mappings between locally convex spaces; Indag. Math. 33(3), (1971), 261-274.

[22] ________ : Generalized Sequence Spaces; Bull. Soc. Math. Belgique 23 (1971), 123-166.

[23] ________ : On the dual of $L(E,F)$; Jour. reine angew. Math. 276 (1975), 170-176.

[24] ________ : On Λ-bases; Jour. Math. Anal. Appl. 53(3), (1976), 508-520.

[25] De Grande-De Kimpe, N. : Criteria for nuclearity in terms of generalized sequence spaces; Arch. der. Math. 28(1977), 644-651.

[26] Djakov, P. and Mityagin, B.S. : Modifed construction of nuclear Fréchet spaces without bases; J. Funct. Anal. 23(1976), 415-433.

[27] Dragilev, M.M. : On regular bases in nuclear spaces, Amer. Math. Soc. Transl. (2), 93 (1970), 61-82.

[28] Dubinsky, E. : Subspaces without bases in nuclear Fréchet spaces; J. Funct. Anal. 26(1977), 121-130.

[29] ___________ : The Structure of Nuclear Fréchet Spaces; Springer Verlag Lecture Notes 720 (1979).

[30] Dubinsky, E. and Mityagin, B.S. : Quotient spaces without bases in nuclear Fréchet spaces; Can. J. Math. 30(6), (1978), 1296-1305.

[31] Dubinsky, E. and Ramanujan, M.S. : $\Lambda_\infty(\alpha)$-bases and nuclear spaces; J. Indian Math. Soc. 36(1972), 333-345.

[32] ___________ : On λ-Nuclearity; Mem. Amer. Math. Soc. No. 128 (1972).

[33] Dynin, A.S. and Mityagin, B.S. : Criterion for nuclearity in terms of approximative dimension; Bull. Acad. Polon. Sci. III, 8(1960), 535-540.

[34] Fehr, E. and Jarchow, H. : Folgenräume als universelle generatoren fur klassen lokal-konvexer räume; Manus. Math. 25 (1978), 209-240.

[35] Fenske, C. and Schock, E. : Über die diametrale dimension von lokalkonvexen Räumen; BMBW-GMD, Bonn, Nr. 10(1969), 13-22.

[36] ___________ : Nuclear spaces of maximal diametral dimension; Comp. Math. 26 (3), (1973), 303-308.

[37] Garling, D.J.H. : The β-and γ-duality; Proc. Camb. Philos. Soc. 63 (1964), 963-981.

[38] Gelfand, I.M. and Vilenkin, N. Ya. : Generalized Functions, Vol. 4; Academic Press (1964).

[39] Gregory, D.A. : Vector-Valued Sequence Spaces; Dissertation, Univ. of Michigan (1967).

[40] Grothendieck, A. : Produits tensoriels topologiques et espaces nucléaires; Mem. Amer. Math. Soc. 16(1955).

[41] Gupta, M. : The generalized spaces $\ell^1(X)$ and $m_0(X)$; Jour. Math. Anal. Appl. 76(1980).

[42] Gupta, M., Kamthan, P.K. and Patterson, J. : Duals of generalized sequence spaces; to appear in Jour. Math. Anal. Appl.

[43] Gupta, M. and Patterson, J. : Matrix transformations and generalized sequence spaces (submitted)

[44] Hogbe-Nlend, H. : Sur les produits tensoriels bornologiques et les espaces a bornologie nucléaire; C.R. Acad. Sci. Paris Ser A-B 268 (1969), A 1602-A1605.

[45] Horváth, J. : Topological Vector Spaces and Distributions, I; Addison Wesley (1966).

[46] Hutton, C.V. : Approximation numbers of bounded linear operators; Thesis, Louisiana State University (1973).

[47] ―――――― : On the approximation numbers of an operator and its adjoint; Math. Ann. 210(1974), 277-280.

[48] Jarchow, H. : Proc. of the International Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics (held in Leipzig, 1977), p. 124.

[49] Joichi, A. : (Z,λ)-nuclear mappings and Z-nuclear spaces; Hiroshima Math. J. 7 (1977), no. 1, 33-59.

[50] Kalton, N.J. : On absolute bases; Math. Ann. 200 (1973), 209-225.

[51] Kamthan, P.K. : Kolmogorov's diameters and their applications; Aligarh Bull. Math. 7 (1977), 9-21.

[52] ______ : A nuclearity criterion for spaces having a Schauder base (submitted).

[53] ______ : λ-nuclear operators and their applications (submitted).

[54] ______ : Behaviour of transformations on sequence spaces (submitted).

[55] Kamthan, P.K. and Gupta, M. : Weak Schauder bases and completeness; Proc. Roy. Irish Acad. 78, A(7), (1978), 51-54.

[56] ______ : Several notions of absolute bases; Jour. reine angew. Math. 307/308 (1979), 160-165.

[57] ______ : Addendum and Corrigendum; Several notions of absolute bases; Jour. reine angew. Math. 317 (1980), p. 220.

[58] ______ : Sequence Spaces and Series; Lecture Notes, 65, Marcel Dekker Inc. New York (1981).

[59] ______ : Bases in Topological Vector Spaces; (monograph under preparation).

[60] Kolmogorov, A.N. : On some asymptotic characterizations of totally bounded metric spaces; DAN U.S.S.R., 108 (1956), 385-388.

[61] ______ : On the linear dimension of topological vector spaces; DAN U.S.S.R., 120 (1958), 239-241.

[62] Kolmogorov, A.N. and Tikhomirov, V.M. : ϵ-entropy and ϵ-capacity of sets in function spaces; (Eng. transl.), Amer. Math. Soc. Transl. 17 (2), (1961), 277-364.

[63] Kōmura, T. and Kōmura, Y. : Über die Einbettung der nuklearen Räumen in $(s)^A$; Math. Ann. 162 (1966), 284-288.

[64] Kōmura, Y. and Koshi, S. : Nuclear vector lattices; Math. Ann. 163 (1966), 105-110.

[65] Köthe, G. : Topological Vector Spaces I; Springer-Verlag, Berlin-Heidelberg-New York (1969).

[66] ____________ : Stark nukleare Folgenräume; J. Fac. Sci. Univ. Tokyo Sect.I 17 (1970), 291-296.

[67] ____________ : Nukleare (F)-und (DF)-Folgenräume; Theory of sets and topology (1972), 327-332.

[68] ____________ : Topological Vector Spaces II; Springer-Verlag, Berlin-Heidelberg-New York (1980).

[69] Lazar, A.J. and Retherford, J.R. : Nuclear spaces, Schauder bases and Choquet simplexes; Pac. J. Math. 37 (2), (1971), 409-419.

[70] Marti, J.T. : Introduction to the Theory of Bases; Springer Verlag, New York Inc. (1969).

[71] Martineau, A. : Sur une propriete universelle de ℓ' espace des distributions de L. Schwartz; C.R. Acad. Sci. Paris 259 (1964), 3162-3164.

[72] Mityagin, B.S. : The connection between the ϵ-entropy, the rapidity of approximation and the nuclearity of a compactum in a linear space; DAN U.S.S.R. 134 (1960), 765-768.

[73] ____________ : Approximative dimension and bases in nuclear spaces; Russ. Math. Surveys, 16(1961), 59-127.

[74] ____________ : Nuclear Fréchet spaces without bases II; Bull. Acad. Polon.Sci. 24 (1976), 475-480.

[75] Mityagin, B.S. and Zobin, N.M. : Contre-example a l´existence d´une base dans une espace de Fréchet nucléaire; C.R. Acad. Paris, Ser A., 279 (1974), 3255-3256; 325-327.

[76] Mori, Y. : An example of a λ(P)-nuclear Fréchet space; Math. Sem. Notes (Kobe Univ.) 5(1977), 289-294.

[77] Moscatelli, V.B. : On the existence of universal λ-nuclear Fréchet spaces; J. reine angew. Math. 301 (1978), 1-26.

[78] ______________ : Fréchet spaces without continuous norms and without bases; Bull. Lond. Math. Soc. 12 (1980), 63-66.

[79] Nelimarkka, E. : On operator ideals and locally convex A-spaces with applications to λ-nuclearity; Ann. Acad. Sci. Fenn. A.I. Math. Diss. 13(1977).

[80] ______________ : On λ(P;ℕ)-nuclearity and operator ideals; to appear in Math. Nachr.

[81] ______________ : The approximation property and locally convex spaces defined by the ideal of approximable operators; Report-HTKK-MAT-A176 (1980), 1-13.

[82] ______________ : On spaces of holomorphic functions on locally convex spaces defined by an operator ideal (preprint).

[83] Pelczyński, A. : On the approximation of S-spaces by finite dimensional spaces; Bull. Acad. Polon. Sci. 5, no. 9 (1957), 879-881.

[84] Persson, A. und Pietsch, A. : p-nukleare und p-integrale Abbildungen in Banachräumen; Stud. Math. 33(1969), 19-62.

[85] Pietsch, A. : Verallgemeinerte Vollkommene Folgenräume; Akademie-Verlag, Berlin (1962).

[86] ______________ : Eine neue Charakterisierung der nuklearen lokalkonvexen Räumen; Math. Nachr. 25 (1963), 31-36, 51-58.

[87] Pietsch, A. : Absolut summierende Abbildungen in lokalkonvexen Räumen; Math. Nachr. 27 (1963), 77-103.

[88] ________ : F-Räume mit absoluter basis; Stud. Math. 26 (1966), 233-238.

[89] ________ : Nuclear Locally Convex Spaces; Springer-Verlag (1972).

[90] ________ : Operator Ideals; North Holland Publishing Company (1980).

[91] Ramanujan, M.S. : Power series spaces $\Lambda(\alpha)$ and associated $\Lambda(\alpha)$-nuclearity; Math. Ann. 189 (1970), 161-168.

[92] ________ : Proc. of the International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics (held in Leipzig, (1977), p. 124.

[93] Ramanujan, M.S. and Rosenberger, B. : On $\lambda(\phi;P)$-nuclearity; Comp. Math. 34(2), (1977), 113-125.

[94] ________ : On the Djakov-Mityagin example of a nuclear Fréchet space without basis; Bull. Acad. Polon. Sci. 26 (3), (1978), 231-236.

[95] Ramanujan, M.S. and Terzioğlu, T. : Power series spaces $\Lambda_k(\alpha)$ of finite type and related nuclearities; Stud. Math. 53 (1975), 1-13.

[96] ________ : Diametral dimension of Cartesian products, stability of smooth sequence spaces and applications; J. reine angew. Math. 280 (1976), 163-171.

[97] Randtke, D. : Characterisation of precompact maps, Schwartz spaces and nuclear spaces; Trans. Amer. Math. Soc. 165 (1972), 87-101.

[98] Robinson, W.B. : On $\Lambda_1(\alpha)$-nuclearity; Duke Math. Jour. 40(3), (1973), 541-546.

[99] Robinson, W.B. : Relationship between λ-nuclearity and pseudo-μ-nuclearity; Trans. Amer. Math. Soc. 201 (1975), 291-303.

[100] Rosenberger, B. : ϕ-nukleare Räume; Math. Nachr. 52 (1972), 147-160.

[101] ______________ : Universal generators for varieties of nulcear spaces; Trans. Amer. Math. Soc. 184 (1973), 275-290.

[102] Rosenberger, B. und Schock, E. : Über nukleare (F)-Räume mit basis; Comp. Math. 25 (1972), 207-219.

[103] Rosier, R.C. : Generalized Sequence Spaces; Dissertation, Univ. of Maryland, U.S.A. (1970).

[104] Rudin, W. : Functional Analysis; Tata McGraw-Hill Publishing Company, New Delhi (1974).

[105] Saxon, S.A. : Embedding nuclear spaces in products of an arbitrary Banach space; Proc. Amer. Math. Soc. 34 (1972), 138-140.

[106] Schatten, R. : The cross space of linear transformations I; Ann. Math., 47 (1946), 73-84.

[107] Schatten, R. : Norm Ideals of Completely Continuous Operators; Springer-Verlag (1960).

[108] Schatten, R. and Neumann, J.von, : The cross space of linear transformations II,III; Ann. Math. 47 (1946), 608-630; 49 (1948), 557-582.

[109] Schock, E. : (F)-Räume mit p-absoluter basis; BMBW-GMD 10(1969), 23-28.

[110] ______________ : Diametrale dimension, approximative dimension und angwendungen; BMBW-GMD, 43 (1971), p. 48.

[111] Shaefer, H.H. : Topological Vector Spaces; Springer-Verlag (1971).

[112] Singer, I. : Bases in Banach Spaces I., Springer Verlag (1970).

[113] Spuhler, P. : Λ-nukleare Räume; Thesis, Univ. of Frankfurt (1970).

[114] Terzioğlu, T. : Die diametrale dimension von lokalkonvexen Räumen Collect. Math. 20 (1), (1969), 49-99.

[115] ____________ : Linear operators on nuclear spaces and their extension properties; Rev. Fac. Sci. Univ. Istanbul 37 (A), (1972), 1-21.

[116] ____________ : Smooth sequence spaces and associated nuclearity; Proc. Amer. Math. Soc. 37 (2), (1973), 497-504.

[117] ____________ : Stability of smooth sequence spaces; Jour. reine angew. Math. 276 (1975); 184-189.

[118] Treves, F. : Topological Vector Spaces, Distributions and Kernels; Academic Press (1967).

[119] Valdivia, M. : Nuclearity and Banach spaces; Proc. Edin. Math. Soc. 20 (1977), 205-209.

[120] Vogt, D. : Subspaces and quotient spaces of s; Functional Analysis, Surveys and Recent Results (Proc. Conf. Paderborn, 1976), 167-187; North Holland 1977.

[121] Waelbroeck, L. : The nuclearity of O(U); Infinite dimensional holomorphy and applications; North Holland Math. Studies 12 (1977), 425-435.

[122] Walker, G. : λ-nuclear and λ-summing operators; Dissertation, Indiana Univ. (1975).

[123] Wong, Y.C. : Schwartz spaces, Nuclear Spaces and Tensor Products; Springer-Verlag Lecture Notes, 726 (1979).

[124] Wojtyński, W. : On conditional bases in non-nuclear Fréchet spaces; Stud. Math. 35 (1970), 77-96.

[125] Zobin, N.M. and Mityagin, B.S. : Examples of nuclear linear metric spaces without a basis; Funct. Anal. and Appl. 8, 4 (1975), 304-313.

VITA

The author was born on May 7, 1954 at Handwara in Kashmir. After having graduated from the University of Kashmir in 1974, he took his Master's degree in 1976 from Muslim University Aligarh and stayed there for one year to undergo further studies in Mathematics. In 1977, he was selected for a research fellowship at the Indian Institute of Technology, Kanpur where he carried out his research under the supervision of Professor P.K. Kamthan. During his Ph.D. programme at the Institute, he took several courses on the theory of nuclear spaces and related topics and also participated in a series of advanced-level seminars on functional analysis. Besides, he has also written some papers on problems related to λ-nuclearity, including the one on 'λ-bases and their applications' which has been accepted for publication in the Journal of Mathematical Analysis and Applications.